OER Secondary Mathematics
EDITION 1

Algebra I

Volume 2

STUDENT EDITION

Acknowledgment

Thank you to all the Texas educators and stakeholders who supported the review process and provided feedback. These materials are the result of the work of numerous individuals, and we are deeply grateful for their contributions.

Notice

These learning resources have been built for Texas students, aligned to the Texas Essential Knowledge and Skills, and are made available pursuant to Chapter 31, Subchapter B-1 of the Texas Education Code.

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Secondary Mathematics

EDITION 1

Algebra I

Course Guide

Welcome to the Course Guide for Secondary Mathematics, Algebra I

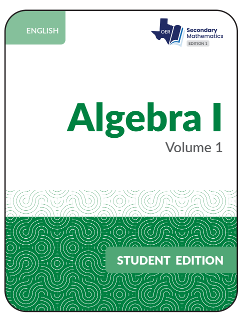
Instructional Design FM-6
Lesson Structure FM-9
Research-Based Strategies FM-15
The Crew FM-17
TEKS Mathematical Process Standards FM-18
Understanding the Problem-Solving Model FM-21
The Problem-Solving Model Graphic Organizer FM-22
Academic Glossary FM-23
What is Productive Struggle? FM-25
Resources for Students and Families FM-26
Course Table of Contents FM-31

INSTRUCTIONAL DESIGN

The instructional materials help you learn math in different ways. There are two types of resources: Learning Together and Learning Individually. These resources provide various learning experiences to develop your understanding of mathematics.

Learning Together

On **Learning Together** days, you spend time engaging in active learning to build mathematical understanding and confidence in sharing ideas, listening to one another, and learning together. The Student Edition is a consumable resource that contains the materials for each lesson.



STUDENT EDITION

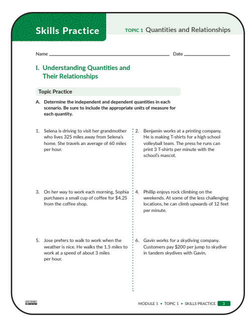
I am a record of your thinking, reasoning, and problem solving.

My lessons allow you to build new knowledge from prior knowledge and experiences, apply math to real-world situations, and learn together in a collaborative classroom.

My purpose is to create mathematical thinkers who are active learners that participate in class.

Learning Individually

On **Learning Individually** days, Skills Practice offers opportunities to engage with skills, concepts, and applications that you learn in each lesson. It also provides opportunities for interleaved practice, which encourages you to flexibly move between individual skills, enhancing connections between concepts to promote long-term learning. This resource will help you build proficiency in specific skills based on your individual academic needs as indicated by monitoring your progress throughout the course.



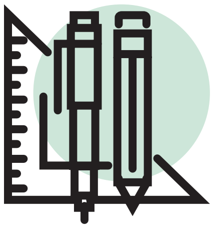
SKILLS PRACTICE

I am targeted practice of each lesson's skills, mathematical concepts, and applications for each topic in the student edition.

My purpose is to provide problem sets for additional practice, enrichment, and extension.

INSTRUCTIONAL DESIGN

The instructional materials intentionally emphasize active learning and sense-making. Deep questions ensure you thoroughly understand the mathematical concepts. The instructional materials guide you to connect related ideas holistically, supporting the integration of your evolving mathematical understanding and developing proficiency with mathematical processes.



Intentional Mathematics Design

Mathematical Coherence: The path through the mathematics develops logically, building understanding by linking ideas within and across grades so you can learn concepts more deeply and apply what you've learned to more complex problems.

TEKS Mathematical Process Standards: The instructional materials support your development of the TEKS mathematical process standards. They encourage you to experiment, think creatively, and test various strategies. These mathematical processes empower you to persevere when presented with complex real-world problems.

Multiple Representations: The instructional materials connect multiple representations of mathematical concepts. Lessons present content visually, algebraically, numerically, and verbally.

Transfer: The instructional materials focus on developing transfer. Doing A and moving on isn't the goal; being able to do A and then do B, C, and D, transferring what you know from A, is the goal.

What principles guide the design and organization of the instructional materials?

Active Learning: Studies show that learning becomes more robust as instruction moves from entirely passive to fully interactive (Chi, 2009). Classroom activities require active learning as students engage by problem solving, explaining and justifying reasoning, classifying, investigate, and analyze peer work.

Discourse Through Collaborative Learning: Collaborative problem solving encourages an interactive instructional model, and we have looked to research to provide practical guidance for making collaboration work. The collaborative activities intentionally promote active dialogue centered on structured activities.

Personalized Learning: Research has proven that problems that capture your interests are more likely to be taken seriously (Baker et. al, 2009). In each lesson, learning is linked to prior knowledge and experiences, creating valuable and authentic opportunities for you to build new understanding on the firm foundation of what you already know. You move from concrete representations and an intuitive understanding of the world to more abstract representations and procedures.

Focus on Problem Solving: Solving problems is an essential life skill that you need to develop. The problem-solving model provides a structure to support you as you analyze and solve problems. It is a strategy you can continue to use as you solve problems in everyday life.

4 Point-Slope Form of a Line

1 OBJECTIVES

- Use the slope formula to derive the point-slope form of a linear equation.
- Construct an equation in point-slope form to model a linear relationship between two quantities.
- Write equations for vertical and horizontal lines.

3 NEW KEY TERM

- point-slope form

2 You have used the slope-intercept form to represent linear relationships. Are there other forms of a linear equation that you can use? How do you write equations for horizontal and vertical lines?

LESSON STRUCTURE

1 Objectives

Objectives are stated for each lesson to help you take ownership of the objectives.

2 Essential Question

Each lesson begins with a statement connecting what you have learned with a question to ponder.

Return to this question at the end of this lesson to gauge your understanding.

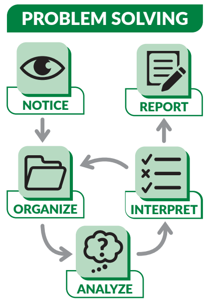
3 New Key Terms

The new mathematical terms for each lesson are identified to help you connect your everyday and mathematical language.

4 Getting Started

Each lesson begins with a Getting Started. When working on the Getting Started, use what you know about the world, what you have learned previously, or your intuition. The goal is just to get you thinking and ready for what's to come.

4 Getting Started



Draining the Pool

Miguel and Nia are pool cleaners who have been hired to drain the community diving pools at the end of the summer. They are comparing the rate at which the three pools drain.

1. For each pool, write an equation in slope-intercept form to represent the linear relationship.

   a. Pool *A* is at a water level of 14 feet and drains at a rate of 3 feet per hour.

   b. Pool *B* is at a water level of 10 feet after draining for 3 hours and drains at a rate of 2 feet per hour.

   c. Pool *C* is at a water level of 15 feet after draining for 2 hours and at 12 feet after draining for 4 hours.

I wonder whether there is a way to make writing the equation of a line more efficient.

2. Compare your process for writing each equation. How are the processes different?

5

ACTIVITY **4.1**

Writing Equations in Point-Slope Form

In the previous lesson, you used the slope, the y-intercept, and the slope formula to write a linear equation. You can also determine the equation of a line without knowing the y-intercept.

WORKED EXAMPLE

To write an equation of a line from a table of values, you can use the slope formula.

x	y
2	6
4	5
6	4

- First, calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 6}{4 - 2}$$
$$= \frac{-1}{2} = -\frac{1}{2}$$

- Next, choose any point from the table.

(2, 6)

- Then, substitute what you know into the slope formula: $m = -\frac{1}{2}$, (2, 6), and the unknown point (x, y).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$

- Finally, rewrite the equation with no variables in a denominator.

$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$
$$-\frac{1}{2}(x - 2) = y - 6$$

The equation is $y - 6 = -\frac{1}{2}(x - 2)$.

This linear equation in the Worked Example is written in *point-slope form*. The **point-slope form** of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is any point on the line.

1. Solve the equation in the Worked Example for y so that the linear equation is in slope-intercept form. What unique information does each form of the linear equation provide? How are they similar?

5 **Activities**

You are going to build a deep understanding of mathematics through a variety of activities in an environment where collaboration and conversations are important and expected.

You will learn how to solve new problems, but you will also learn why those strategies work and how they are connected to other strategies you already know.

Remember:

- It's not just about getting the answer. The process is important.
- Making mistakes is a critical part of learning, so take risks.
- There is often more than one way to solve a problem.

Activities may include real-world problems, sorting activities, Worked Examples, or analyzing sample student work.

Be prepared to share your solutions and methods with your classmates.

6 Talk the Talk

Talk the Talk gives you an opportunity to reflect on the main ideas of the lesson.

- Be honest with yourself.
- Ask questions to clarify anything you don't understand.
- Show what you know!

Don't forget to revisit the question posed on the lesson opening page to gauge your understanding.

Talk the Talk 6

Say What?

You have learned about two forms of a linear equation: the slope-intercept form, $y = mx + b$, and the point-slope form, $y - y_1 = m(x - x_1)$.

1. What information can you determine about each line by looking at the structure of the equation?

   a. $y = \frac{3}{5}x - 4$

   b. $y - 6 = 2(x + 1)$

   c. $y + 4 = 2(x - 0)$

   d. $y = -\frac{2}{7}x$

   e. $y + 5 = -(x - 4)$

   f. $y = 19$

2. Create a context that represents a linear relationship that passes through the point (2, 56) and has an increasing slope. Then, write the equation of the line in point-slope form and slope-intercept form.

Lesson 4 Assignment

7 Write

Compare the slope-intercept and point-slope forms of a linear equation.

Remember 8

The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line. The slope of a horizontal line is 0. The slope of a vertical line is undefined.

9 Practice

Write an equation in point-slope form.

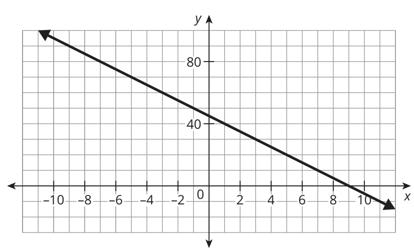
1. $m = 2$; (5, 6)
2. $m = -9.2$; (−17, 10)
3. (−2, −3) and (8, −8)
4. (79, 52) and (−87, 550)
5. A photography studio charges $50 for a sitting fee and 6 prints. Luigi increased his order to 11 prints and paid $65.
6. Lucia is taking the stairs in her building from her floor to the top of the building. After 2 minutes, she was 100 steps from the bottom floor. After 5 minutes, she was 196 steps from the bottom floor.

Write an equation in any form.

7. A newspaper charges a flat fee plus a charge per day to place a classified ad.

Number of Days	Total Charge ($)
2	8.00
4	13.00
6	18.00

8.



ASSIGNMENT

7 Write

Reflect on your work and clarify your thinking.

8 Remember

Take note of the key concepts from the lesson.

9 Practice

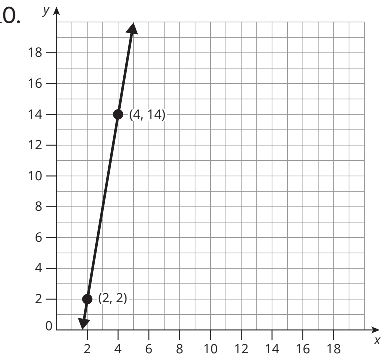
Use the concepts learned in the lesson to solve problems.

Lesson 4 Assignment

9.

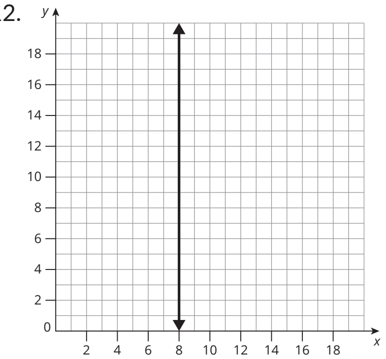
x	y
−10	50
−2	10
4	−20
14	−70

10.



11. Xavier is traveling on a toll road. He plans to exit the road 5 miles ahead and pay \$1.75. He changes his plans and travels 9 miles and pays \$2.75.

12.



10 Prepare

Solve each equation for y.

1. $-2y = -x + 7$
2. $\frac{3}{4}y = x - 6$
3. $2x + 3y = 6$
4. $\frac{1}{2}x - 4y = 8$

ASSIGNMENT

10 Prepare

Get ready for the next lesson.

WORKED EXAMPLE

Consider the sequence generated using $a_n = a_{n-1} + (-2)$ where $a_1 = 11$ and n is a whole number greater than 1.

a_1 represents the first term of the sequence and a_n represents the n^{th} term of the sequence.

Since I know the first term of the sequence, to determine the second term a_2, I add -2 to 11.

$a_2 = a_{2-1} + -2 = a_1 + -2 = 11 + -2 = 9$

I can determine the 3rd and 4th term of the sequence by continuing the pattern.

$a_3 = a_{3-1} + -2 = a_2 + -2 = 9 + -2 = 7$

$a_4 = a_{4-1} + -2 = a_3 + -2 = 7 + -2 = 5$

The sequence is 11, 9, 7, 5. . .

The pattern is to add the same negative number, -2, to each term to determine the next term.

add −2   add −2   add −2

Sequence: 11, 9, 7, 5, ...

This sequence is arithmetic and the common difference d is -2.

Ask Yourself . . .

- What is the main idea?
- How would this work if I changed the numbers?
- Have I used these strategies before?

WORKED EXAMPLE

When you see a Worked Example:

- Take your time to read through it.
- Question your own understanding.
- Think about the connections between steps.

THUMBS UP

When you see a Thumbs Up icon:

- Take your time to read through the correct solution.
- Think about the connections between steps.

THUMBS DOWN

When you see a Thumbs Down icon:

- Take your time to read through the incorrect solution.
- Think about what error was made.

WHO'S CORRECT?

When you see a Who's Correct icon:

- Take your time to read through the situation.
- Question the strategy or reason given.
- Determine whether the work is correct or incorrect.

Maria and Abby each convert the given formula to degrees Fahrenheit.

Maria

$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - \frac{160}{9}$$
$$9(C) = 9\left(\frac{5}{9}F - \frac{160}{9}\right)$$
$$9C = 5F - 160$$
$$9C + 160 = 5F$$
$$\frac{9C}{5} + \frac{160}{5} = \frac{5F}{5}$$
$$\frac{9}{5}C + 32 = F$$

Abby

$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - 32$$
$$9(C) = 9\left(\frac{5}{9}F - 32\right)$$
$$9C = 5F - 288$$
$$9C + 288 = 5F$$
$$\frac{9C}{5} + \frac{288}{5} = \frac{5F}{5}$$
$$\frac{9}{5}C + 576 = F$$

Ask Yourself . . .

- Why is this method correct?
- Have I used this method before?

Ask Yourself . . .

- Where is the error?
- Why is it an error?
- How can I correct it?

4. Consider a sequence in which the first term is 64 and each term after that is calculated by dividing the previous term by 4. Margaret says that this sequence ends at 1 because there are no whole numbers that come after 1. Jasmine disagrees and says that the sequence continues beyond 1. Who is correct? When Margaret is correct, explain why. When Jasmine is correct, predict the next two terms of the sequence.

Ask Yourself . . .

- Does the reasoning make sense?
- If the reasoning makes sense, what is the justification?
- If the reasoning does not make sense, what error was made?

The Crew

The Crew is here to help you on your journey. Sometimes, they will remind you about things you already learned. Sometimes, they will ask you questions to help you think about different strategies. Sometimes, they will share fun facts. They are members of your group—someone you can rely on!

Teacher aides will guide you along your journey. They will help you make connections and remind you to think about the details.

TEKS Mathematical Process Standards

Effective communication and collaboration are essential skills of a successful learner. With practice, you can develop the habits of mind of a productive mathematical thinker. The "I Can" expectations listed below align with the TEKS mathematical process standards and encourage students to develop their mathematical learning and understanding.

Apply mathematics to problems arising in everyday life, society, and the workplace.

I CAN:

- use the mathematics that I learn to solve real-world problems.
- interpret mathematical results in the contexts of a variety of problem situations.

Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying a solution, and evaluating the problem solving process and reasonableness of the solution.

I CAN:

- explain what a problem "means" in my own words.
- create a plan and change it if necessary.
- ask useful questions in an attempt to understand the problem.
- explain my reasoning and defend my solution.
- reflect on whether my results make sense.

Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate; and techniques including mental math, estimation, and number sense as appropriate, to solve problems.

I CAN:

- use a variety of different tools that I have to solve problems.
- recognize when a tool that I have to solve problems might be helpful and when it has limitations.
- look for efficient methods to solve problems.
- estimate before I begin calculations to inform my reasoning.

Communicate mathematical ideas, reasoning, and their implications using multiple representations including symbols, diagrams, graphs, and language, as appropriate.

I CAN:

- communicate and defend my own mathematical understanding using examples, models, or diagrams.
- use appropriate mathematical vocabulary in communicating mathematical ideas.
- make generalizations based on results.
- apply mathematical ideas to solve problems.
- interpret my results in terms of various problem situations.

Create and use representations to organize, record, and communicate mathematical ideas.

I CAN:

- consider the units of measure involved in a problem.
- label diagrams and figures appropriately to clarify the meaning of different representations.
- create an understandable representation of a problem situation.

Analyze mathematical relationships to connect and communicate mathematical ideas.

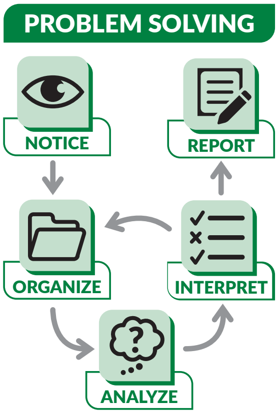
I CAN:

- identify important relationships in a problem situation.
- use what I know to solve new problems.
- analyze and organize information.
- look closely to identify patterns or structure.
- look for general methods and more efficient ways to solve problems.

Display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication.

I CAN:

- work carefully and check my work.
- distinguish correct reasoning from reasoning that is flawed.
- use appropriate mathematical vocabulary when I talk with my classmates, my teacher, and others.
- specify the appropriate units of measure when I explain my reasoning.
- calculate accurately and communicate precisely to others.



Understanding the Problem-Solving Model

Productive mathematical thinkers are problem solvers. These instructional materials include a problem-solving model to help you develop proficiency with the TEKS mathematical process standards. An opportunity to use the problem-solving model is present every time you see the problem-solving model icon.

The problem-solving model represents a strategy you can use to make sense of problems you must solve. The model provides questions you can use to guide your thinking and the graphic organizer provides a place for you to show and organizer your work.

Using the Problem-Solving Model

Notice | Wonder

Understand the situation by asking these questions.

- What do I notice?
- What do I wonder?
- How do I analyze the given information to identify what is important?
- Do I have enough information to formulate a plan and determine a solution?

Organize | Mathematize

Devise a plan for your mathematical approach by asking these questions.

- What mathematical relationships exist between this problem and similar problems I have solved?
- What plan or strategy can I use to solve this problem?
- How can I efficiently solve this problem?
- How can I organize, record, and communicate my mathematics?

Predict | Analyze

Carry out your plan to determine a solution. Then, ask yourself the following questions.

- Did I display my work using multiple representations?
- Did I explain my reasoning in terms of the problem situation?
- Did I communicate the strategy used to determine the solution?
- Did I justify my mathematical argument clearly using precise mathematical language?
- Can I use my mathematical reasoning to make any predictions?

Test | Interpret

Look back at your work and ask these questions.

- Does my solution clearly and completely answer the original question/problem?
- Is my solution reasonable?
- Does my solution make sense in terms of the problem situation?
- Can I solve the problem using a different strategy? Would another strategy be more efficient?
- Can I justify my solution?

Report

As you share your mathematical reasoning with others, ask these questions.

- Did you use multiple representations to represent your mathematics?
- Did you justify your mathematical reasoning?
- Can others understand my process and solution?

The Problem-Solving Model Graphic Organizer

NOTICE **Understand the Problem**	ORGANIZE **Devise a Plan**
PREDICT **Carry Out the Plan**	
INTERPRET **Look Back**	
REPORT **Report**	

Academic Glossary

Knowing what these terms mean and using them will help you demonstrate proficiency with the TEKS mathematical process standards as you think, reason, and communicate your ideas.

Analyze

Definition

Study or look closely for patterns. Analyzing can involve examining or breaking a concept down into smaller parts to gain a better understanding of it.

Ask Yourself

- Do I see any patterns?
- Have I seen something like this before?
- What happens if the shape, representation, or numbers change?

- Examine
- Evaluate
- Determine
- Observe
- Consider
- Investigate
- What do you notice?
- What do you think?
- Sort and match

Explain Your Reasoning

Definition

Give details or describe how to determine an answer or solution. Explaining your reasoning helps justify conclusions.

Ask Yourself

- How should I organize my thoughts?
- Is my explanation logical?
- Does my reasoning make sense?
- How can I justify my answer to others?

- Show your work
- Explain your calculation
- Justify
- Why or why not?

- Show
- Sketch
- Draw
- Create
- Plot
- Graph
- Write an equation
- Complete the table

Represent

Definition

Display information in various ways. Representing mathematics can be done using words, tables, graphs, or symbols.

Ask Yourself

- How should I organize my thoughts?
- How do I use this model to show a concept or idea?
- What does this representation tell me?
- Is my representation accurate?

- Predict
- Approximate
- Expect
- About how much?

Estimate

Definition

Make an educated guess based on the analysis of given data. Estimating first helps inform reasoning.

Ask Yourself

- Does my reasoning make sense?
- Is my solution close to my estimation?

- Demonstrate
- Label
- Display
- Compare
- Determine
- Define
- What are the advantages?
- What are the disadvantages?
- What is similar?
- What is different?

Describe

Definition

Represent or give an account of in words. Describing communicates mathematical ideas to others.

Ask Yourself

- How should I organize my thoughts?
- Is my explanation logical?
- Did I consider the context of the situation?
- Does my reasoning make sense?

What is Productive Struggle?

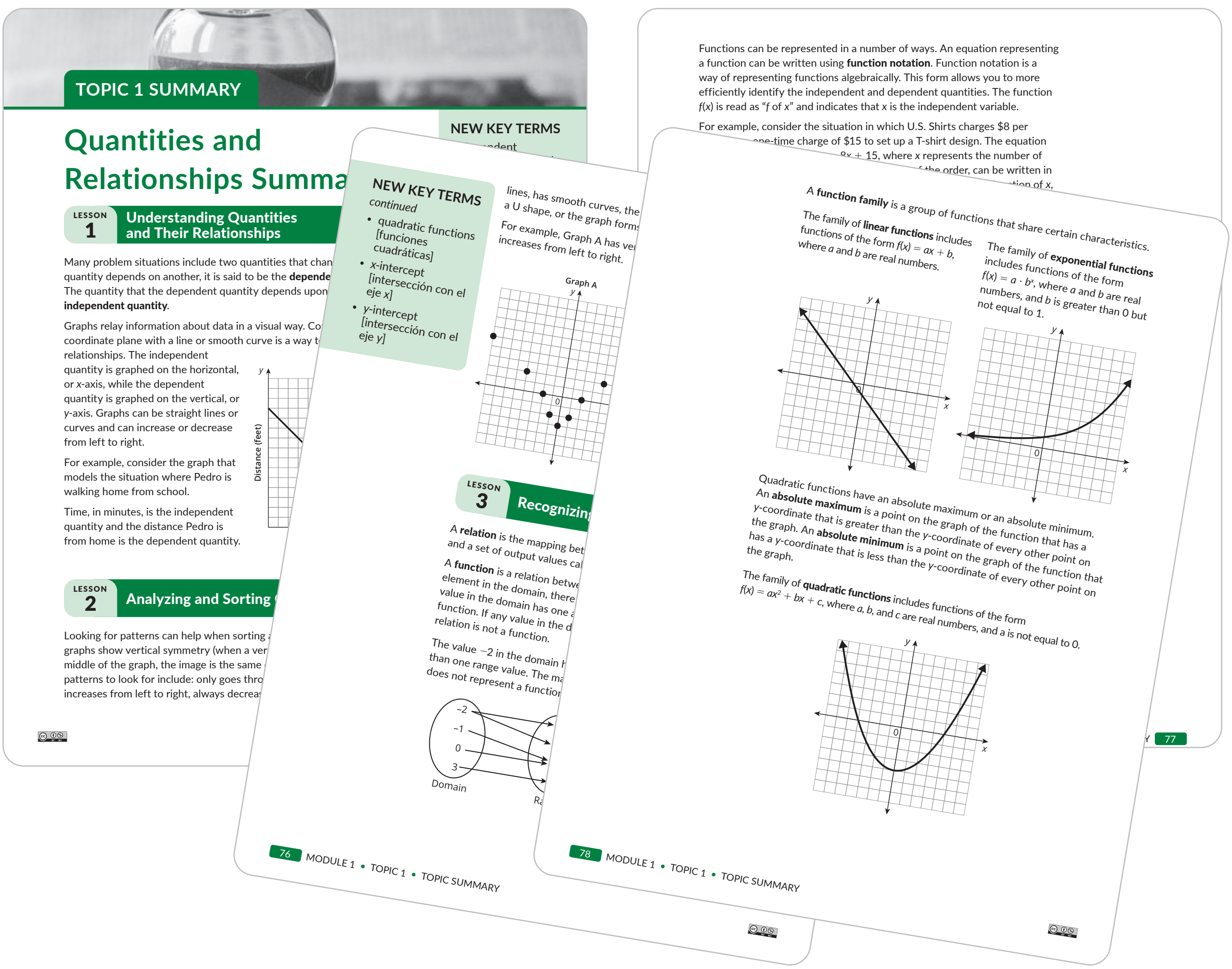
Imagine you are trying to beat a game or solve a puzzle. At some point, you are not sure what to do next. You don't give up; you keep trying repeatedly until you accomplish your goal. You call this process *productive struggle*. Productive struggle is when you choose to persist, think creatively, and try various strategies to accomplish your goal. Productive struggle supports you as you develop characteristics and habits that you will use in everyday life.

Things to do:	Things not to do:
• Persevere. • Think creatively. • Try different strategies. • Look for connections to other questions or ideas. • Ask questions that help you understand the problem. • Help your classmates without telling them the answers.	• Get discouraged. • Stop after trying your first attempt. • Focus on the final answer. • Think you have to make sense of the problem on your own.

This course will challenge you by providing you with opportunities to solve problems that apply the mathematics you know to the real-world. As you work through these problems, your first approach may not work. This is okay, even when your initial strategy does not work, you are learning. In addition, you will discover multiple ways to solve a problem. Looking at various strategies will help you evaluate the problem-solving process and identify an efficient approach. As you persist in problem-solving, you will better understand the math.

Topic Summary

Each topic includes a Topic Summary. The Topic Summary contains a list of all new key terms addressed in the topic and a summary of each lesson, including worked examples and new key term definitions. Use the Topic Summary to review each lesson's major concepts and strategies as you complete assignments and/or share your learning outside of class.



Topic Self-Reflection

The Topic Self-Reflection, provided at the end of each topic, empowers you to develop confidence in your mathematical understanding and monitor your own learning processes. Taking the time for self-reflection helps you identity your strengths and where you want to focus efforts to improve.

Use the Topic Self Reflection throughout the topic to monitor your progress toward the mathematical goals for the topic.

TOPIC 1 SELF-REFLECTION Name: ____________________

Quantities and Relationships

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Quantities and Relationships* topic by:

TOPIC 1: *Quantities and Relationships*	Beginning of Topic	Middle of Topic	End of Topic
choosing appropriate scale and origin for graphs.	○	○	○
identifying the appropriate unit of measure for each variable or quantity.	○	○	○
analyzing a graph and stating the key characteristics of the graph.	○	○	○
using a problem situation to explain what the key features of a graph mean in real-world context.	○	○	○
deciding whether relations represented verbally, tabularly, graphically, and symbolically define a function.	○	○	○
recognizing a linear, exponential, or quadratic function by its equation or graph.	○	○	○
evaluating functions, expressed in function notation, given one or more elements in their domain.	○	○	○
determining the domain and range and the independent and dependent quantities in a relationship.	○	○	○

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

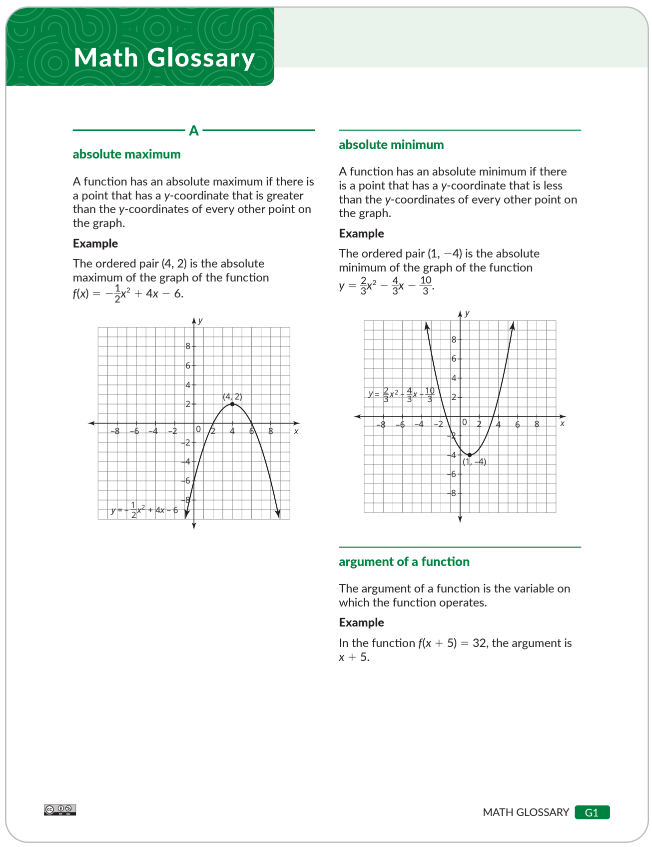
1. Describe a new strategy you learned in the *Quantities and Relationships* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Math Glossary

A course-specific math glossary is available to utilize and reference while you are learning. Use the glossary to locate definitions and examples of math key terms.


Math Glossary

A

absolute maximum

A function has an absolute maximum if there is a point that has a *y*-coordinate that is greater than the *y*-coordinates of every other point on the graph.

Example

The ordered pair (4, 2) is the absolute maximum of the graph of the function $f(x) = -\frac{1}{2}x^2 + 4x - 6$.

absolute minimum

A function has an absolute minimum if there is a point that has a *y*-coordinate that is less than the *y*-coordinates of every other point on the graph.

Example

The ordered pair (1, −4) is the absolute minimum of the graph of the function $y = \frac{2}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$.

argument of a function

The argument of a function is the variable on which the function operates.

Example

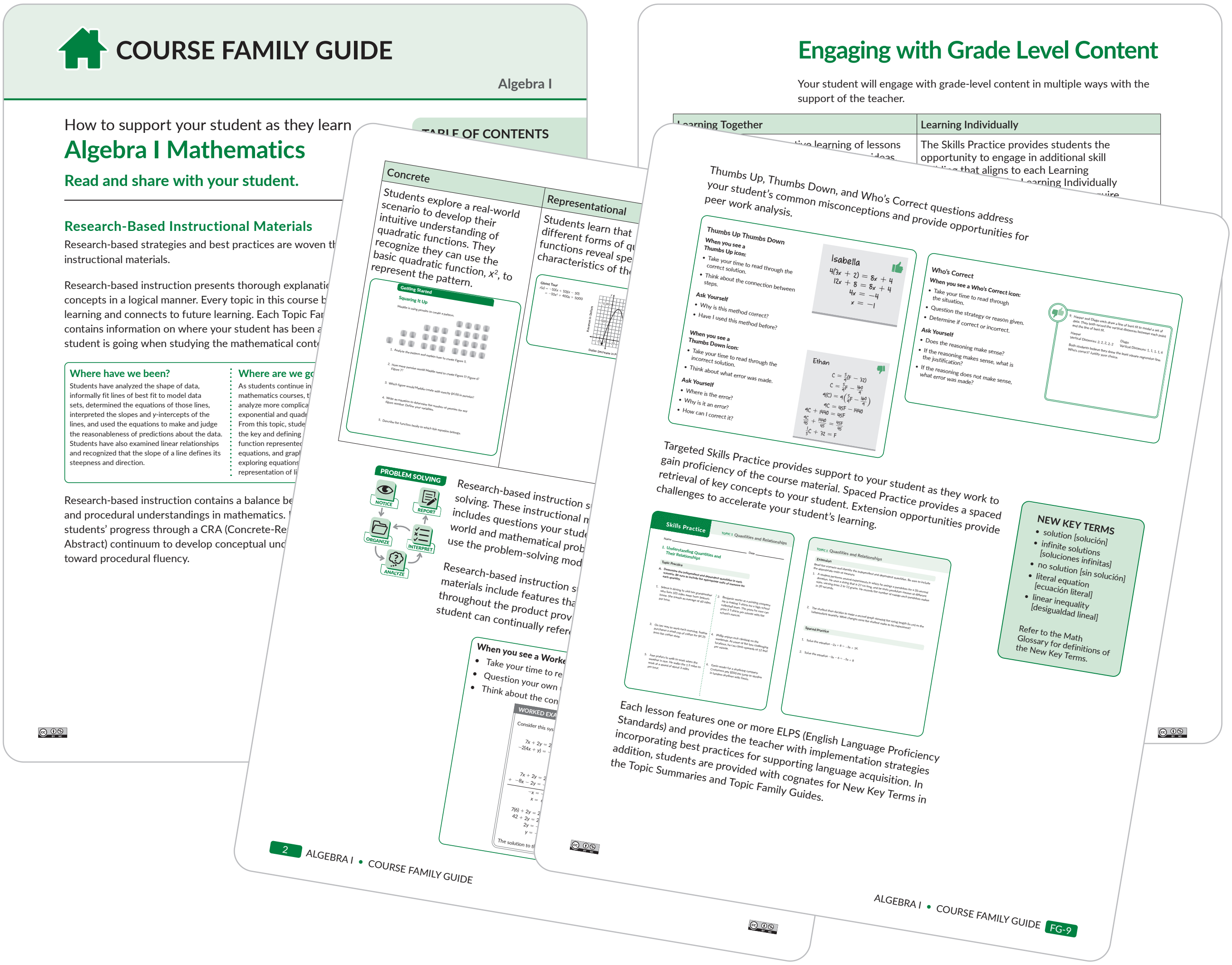
In the function $f(x + 5) = 32$, the argument is $x + 5$.

MATH GLOSSARY G1

Course Family Guide

The Course Family Guide provides you and your family an overview of the course design. The guide details the resources available to support your learning, such as the Math Glossary, the Topic Family Guide, the Topic Self-Reflection, and the Topic Summaries.

The purpose of the Course Family Guide is to bridge your learning in the classroom to your learning at home. The goal is to empower you and your family to understand the concepts and skills learned in the classroom so that you can review, discuss, and solidify the understanding of these key concepts together.



Topic Family Guides

Each topic contains a Topic Family Guide that provides an overview of the math of the topic and answers the questions, "Where have we been?" and "Where are we going?" Additional components of the Topic Family Guide are an example of a math model or strategy taught in the topic, definitions of key terms, busting of a math myth, and questions family members can ask you to support your learning.

Learning outside of the classroom is crucial to student success at school. The Topic Family Guide serves to assist families in talking to students about the learning that is happening in the classroom.

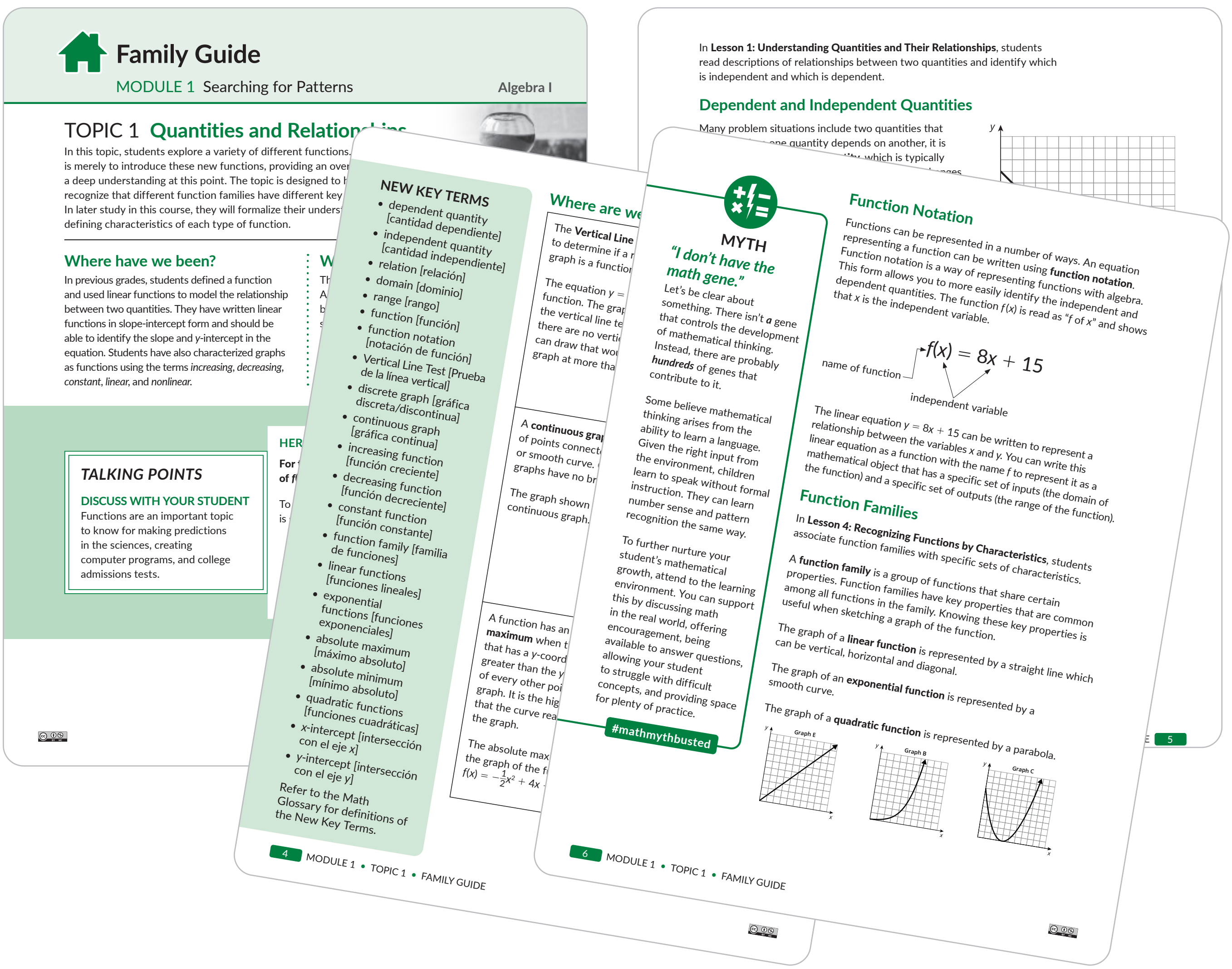


Table of Contents

MODULE 1 Searching for Patterns

TOPIC 1 Quantities and Relationships 3

Introduction to the Problem-Solving Model and Learning Resources

LESSON 1 Understanding Quantities and Their Relationships

LESSON 2 Analyzing and Sorting Graphs

LESSON 3 Recognizing Functions and Function Families

LESSON 4 Recognizing Functions by Characteristics

TOPIC 2 Sequences 81

LESSON 1 Recognizing Patterns and Sequences

LESSON 2 Arithmetic and Geometric Sequences

LESSON 3 Determining Recursive and Explicit Expressions from Contexts

MODULE 2 Exploring Constant Change

TOPIC 1 Linear Functions 145

LESSON 1 Least Squares Regression

LESSON 2 Correlation

LESSON 3 Making Connections Between Arithmetic Sequences and Linear Functions

LESSON 4 Point-Slope Form of a Line

LESSON 5 Using Linear Equations

LESSON 6 Making Sense of Different Representations of a Linear Function

TOPIC 2 Transforming and Comparing Linear Functions . . 267

LESSON 1 Transforming Linear Functions

LESSON 2 Vertical and Horizontal Transformations of Linear Functions

LESSON 3 Determining Slopes of Perpendicular Lines

LESSON 4 Comparing Linear Functions in Different Forms

MODULE 3 Modeling Linear Equations and Inequalities

TOPIC 1 Linear Equations and Inequalities 343

LESSON 1 Solving Linear Equations

LESSON 2 Literal Equations

LESSON 3 Modeling Linear Inequalities

TOPIC 2 Systems of Linear Equations and Inequalities . . .391

LESSON 1 Using Graphing to Solve Systems of Equations

LESSON 2 Using Substitution to Solve Linear Systems

LESSON 3 Using Linear Combinations to Solve a System of Linear Equations

LESSON 4 Graphing Inequalities in Two Variables

LESSON 5 Systems of Linear Inequalities

LESSON 6 Solving Systems of Equations and Inequalities

MODULE 4 Investigating Growth and Decay

TOPIC 1 Introduction to Exponential Functions 509

LESSON 1 Properties of Powers with Integer Exponents

LESSON 2 Analyzing Properties of Powers

LESSON 3 Geometric Sequences and Exponential Functions

LESSON 4 Rewriting Square Roots

LESSON 5 Rational Exponents and Graphs of Exponential Functions

TOPIC 2 Using Exponential Equations 607

LESSON 1 Exponential Equations for Growth and Decay

LESSON 2 Interpreting Parameters in Context

LESSON 3 Modeling Using Exponential Functions

MODULE 5 Maximizing and Minimizing

TOPIC 1 Introduction to Quadratic Functions 657

LESSON 1 Exploring Quadratic Functions

LESSON 2 Key Characteristics of Quadratic Functions

LESSON 3 Quadratic Function Transformations

LESSON 4 Horizontal Transformations and Vertex Form

TOPIC 2 Polynomial Operations 757

LESSON 1 Adding and Subtracting Polynomials

LESSON 2 Multiplying Polynomials

LESSON 3 Polynomial Division

TOPIC 3 Solving Quadratic Equations 825

LESSON 1 Representing Solutions to Quadratic Equations

LESSON 2 Solutions to Quadratic Equations in Vertex Form

LESSON 3 Factoring and Completing the Square

LESSON 4 The Quadratic Formula

LESSON 5 Using Quadratic Functions to Model Data

Math Glossary G-1

MODULE 4

Investigating Growth and Decay

TOPIC 1 Introduction to Exponential Functions **509**

TOPIC 2 Using Exponential Equations **607**

ALGEBRA I

TOPIC 1

You can recognize the graph of an exponential equation because it increasingly increases (or decreasingly decreases).

Introduction to Exponential Functions

LESSON 1 Properties of Powers with Integer Exponents 511

LESSON 2 Analyzing Properties of Powers 537

LESSON 3 Geometric Sequences and Exponential Functions . . 551

LESSON 4 Rewriting Square Roots 567

LESSON 5 Rational Exponents and Graphs of Exponential Functions 579

1 Properties of Powers with Integer Exponents

OBJECTIVES

- Expand a power into a product.
- Write a product as a power.
- Simplify numeric and algebraic expressions containing integer exponents.
- Develop rules to simplify a product of powers, a power of a power, and a quotient of powers.
- Apply the properties of integer exponents to create equivalent expressions.

NEW KEY TERMS

- power
- base
- exponent

You have learned how to evaluate numeric expressions involving whole-number exponents.

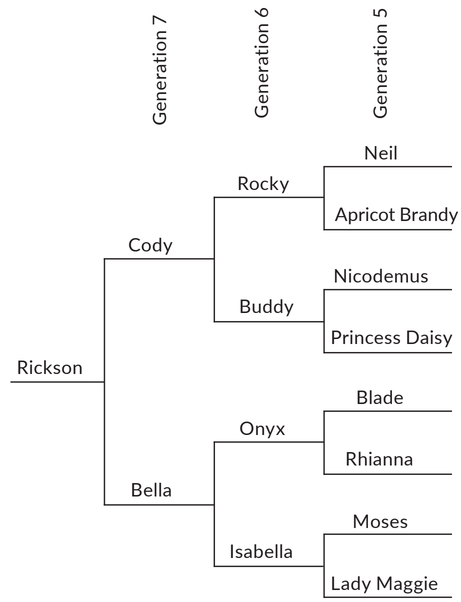
In this lesson, you will develop the properties of integer exponents to generate equivalent numeric and algebraic expressions.

How can you use the properties of integer exponents to generate equivalent numeric expressions?

Getting Started

Three Generations

Lucas adopted an eighth-generation purebred English Mastiff puppy that he named Rickson. The breeder provided documentation that verified a portion of Rickson's lineage going back three generations, as shown.



Ask Yourself . . .
What patterns do you notice?

A dog's lineage is similar to a person's family tree. It shows a dog's parents, grandparents, and great-grandparents.

1. How many parents does Rickson have? What are his parents' names?

2. How many grandparents does Rickson have?

3. How many great-grandparents does Rickson have?

4. What pattern is there in the number of dogs in each generation?

ACTIVITY
1.1 Review of Powers and Exponents

Lucas wants to trace Rickson's lineage back through all eight generations. How many sires (male parents) and dams (female parents) are there in each generation of Rickson's lineage?

1. Complete the second column of the table, Number of Sires and Dams, to show the total number of dogs in each generation.

Generation	Number of Sires and Dams		
Generation 7			
Generation 6			
Generation 5			
Generation 4			
Generation 3			
Generation 2			
Generation 1			

Ask Yourself . . .
Why is the eighth generation not included as part of the table? Which dog is included in the eighth generation?

An expression used to represent the product of a repeated multiplication is a *power*. A **power** has a *base* and an *exponent*. The **base** of a power is the expression that is used as a factor in the repeated multiplication. The **exponent** of a power is the number of times that the base is used as a factor in the repeated multiplication.

$\text{base}^{\text{exponent}}$ ← **power**

WORKED EXAMPLE

You can write a power as a product by writing out the repeated multiplication.

$$2^7 = (2)(2)(2)(2)(2)(2)(2)$$

The power 2^7 can be read as:

- "two to the seventh power."
- "the seventh power of two."
- "two raised to the seventh power."

Think About . . .
How can I write the number of dogs in each generation as a repeated multiplication?

2. Label the third column *Expanded Notation*. Then, write each generation total as a product.

3. Label the fourth column of the table *Power*. Then, write each generation total as a power.

Ask Yourself . . .
How many generations are there before this dog in its lineage?

4. Suppose another dog is a thirteenth-generation purebred. How many dogs are in the first generation? Write your answer as a power and then use a calculator to determine the total number of dogs.

5. How many total sires and dams are there in all three generations shown in Rickson's lineage? Explain your calculation.

ACTIVITY 1.2

Practice with Powers

In this activity, you will investigate the role of parentheses in expressions containing exponents.

1. Identify the base(s) and exponent(s) in each. Then, write each power as a product. Finally, evaluate the power.

   a. 5^3

   b. $(-9)^5$

   c. -11^3

   d. $(4)^5(3)^6$

Think About . . .
When the negative sign is not in parentheses, it's not part of the base. Instead, there is a factor of −1.

2. Write each as a product. Then, calculate the product.

   a. -1^2

   b. -1^3

   c. -1^4

   d. -1^5

   e. $(-1)^2$

   f. $(-1)^3$

   g. $(-1)^4$

   h. $(-1)^5$

3. Consider Questions 2(f) and 2(h). What conclusion can you draw about a negative number raised to an odd power?

4. Consider Questions 2(e) and 2(g). What conclusion can you draw about a negative number raised to an even power?

ACTIVITY 1.3 Multiplying and Dividing Powers

1 gigabyte = 1024 megabytes

1 megabyte = 1024 kilobytes

1 kilobyte = 1024 bytes

File sizes of eBooks, podcasts, and song downloads depend on the complexity of the content and the number of images.

WORKED EXAMPLE

Suppose that a medium-sized eBook contains about 1 megabyte (MB) of information.

Since 1 megabyte is 1024 kilobytes (kB), and 1 kilobyte is 1024 bytes (B), you can multiply to determine the number of bytes in the eBook:

$$1 \text{ MB} = (1024 \text{ kB})\left(\frac{1024 \text{ B}}{1 \text{ kB}}\right) = 1{,}048{,}576 \text{ B}$$

There are 1,048,576 bytes in the eBook.

Remember . . .
Be sure to use units in your calculations.

1. One model of an eBook can store up to 256 MB of data. A USB jump drive can hold 2 GB of storage. Use the method shown in the Worked Example to calculate each.

   a. Calculate the number of bytes the eBook can store.

   256 MB · ________ · _________ = __________________

   b. A USB jump drive can hold 2 GB of storage. How many bytes can the USB jump drive hold?

   2 GB · _______ · ______ · _________ = __________________

   c. How many times more storage space does the jump drive have than the eBook? Show your work.

WORKED EXAMPLE

Computers use binary math, or the base-2 system, instead of the base-10 system.

Base 10	Base 2
$10^1 = 10$	$2^1 = 2$
$10^2 = (10)(10) = 100$	$2^2 = (2)(2) = 4$
$10^3 = (10)(10)(10) = 1000$	$2^3 = (2)(2)(2) = 8$

2. Revisit Question 1, parts (a) through (c), by rewriting each factor and either your product or quotient as a power of 2.

   a.

   b.

   c.

3. Analyze your answers to Question 2. What do you notice about all the bases in Question 2?

4. In parts (a) and (b), how does the exponent in each product relate to the exponents in the factors?

5. In part (c), how does the exponent in the quotient relate to the exponents in the numerator and denominator?

ACTIVITY 1.4

Product of Powers

In this activity, you will explore different expressions to develop rules to evaluate powers.

1. Rewrite each expression as a product using expanded notation. Then, identify the base or bases and record the number of times the base is used as a factor.

   a. $2^4 \cdot 2^3$

   b. $(-3)^3(-3)^3$

   c. $(4)(4^5)$

   d. $(5^2)(6^2)(5^3)(6)$

   e. $(9^3)(4^2)(9^2)(4^5)$

2. Rewrite each of your answers from Question 1 as a power or a product of powers.

3. What relationship do you notice between the exponents in the original expression and the number of factors?

4. Write a rule that you can use to multiply powers.

5. Use your rule to write an equivalent expression as a power or product of powers in simplest form.

a. $5^2 \cdot 5^3$

b. $x^4 \cdot x^6$

c. $(-3)^3(-3)^5$

d. $(-a)^2(-a)$

e. $(6)(4^7)(6^8)$

f. $(c^3)(d^9)(c^{12})(d^4)$

A power can also be raised to a power.

WORKED EXAMPLE

The exponential expression $(4^2)^3$ is a power of a power. It can be written as two repeated multiplication expressions using the definition of a power.

$$(4^2)^3 = (4^2)(4^2)(4^2)$$

$$= (4 \cdot 4)\cdot(4 \cdot 4)\cdot(4 \cdot 4)$$

There are 6 factors of 4.

6. Use the definition of a power to write repeated multiplication expressions for each power of a power, as modeled in the Worked Example. Then, record the number of factors.

   a. $(8^2)^3$

   b. $(5^4)^2$

   c. $-(6^1)^6$

   d. $((-6)^2)^2$

7. What relationship do you notice between the exponents in each expression in Question 5 and the number of factors? Write each expression as a single power.

8. Write a rule that you can use to determine a power of a power.

9. Write an equivalent expression in simplest form using the rules that you wrote.

a. $6^4 \cdot 6^3$

b. $y^7 \cdot y^8$

c. $(4^3)^5$

d. $(w^7)^3$

e. $r^5 \cdot r^2 \cdot r$

f. $((2)(3))^4$

10. Mason says that $2^6 = 12$. Luna says that $2^6 = 64$. Who is correct? Explain your reasoning.

11. Nahimana says that $2^2 + 2^3 = 2^5$, and Sebastian says that $2^2 + 2^3 \neq 2^5$. Who is correct? Explain your reasoning.

ACTIVITY
1.5 Quotient of Powers

Now, let's investigate what happens when you divide powers with like bases.

1. Write each numerator and denominator as a product. Then, write an equivalent expression in simplest form using exponents.

   a. $\frac{9^5}{9^2}$

   b. $\frac{5^6}{5^3}$

   c. $\frac{x^8}{x^6}$

   d. $\frac{10^2}{10}$

2. What relationship do you notice between the exponents in the numerator and denominator and the exponents in the simplified expression?

3. Write a rule that you can use to divide with powers.

4. Write an equivalent expression in simplest form using the rule that you wrote for a quotient of powers.

   a. $\frac{6^8}{6^3}$

   b. $-\frac{t^7}{t^5}$

   c. $\frac{2^3}{3^2}$

ACTIVITY **1.6**

Powers Equal to 1 and Numbers Less Than 1

You know that any number divided by itself is 1. How can you use that knowledge to develop another rule to evaluate powers?

Consider each representation of 1.

$\frac{4}{4} = 1$ $\frac{9}{9} = 1$ $\frac{25}{25} = 1$ $\frac{x}{x} = 1$

1. Rewrite the numerator and denominator of each fraction as a power. Do not simplify.

2. Next, simplify the fractions you just wrote using the quotient of powers rule. Leave your answer as a power. What do you notice?

An exception is that 0^0 is not equal to 1, because that would mean that using zero as a factor zero times would give you 1, and that's not possible.

3. Write a rule that you can use when raising any base to the zero power.

Let's determine how to use powers to represent numbers that are less than 1.

You know that you can use powers to represent numbers that are greater than or equal to 1.

4. Let's start with 1 and multiply by 10 three times.

   a. Complete the representation. Write each as a power.

$1 = 10^0$

Multiply by 10 = ______________ = ______________

Multiply by 10 = ______________ = ______________

Multiply by 10 = ______________ = ______________

b. Describe what happens to the exponents as the number becomes greater.

5. Now, let's start with 1 and divide by 10 three times.

a. Complete the representation. Write the division as a fraction and then rewrite using the definition of powers. Next, apply the quotient of powers rule, and finally, simplify each expression.

	1	$= \frac{10^0}{10^0}$	$= 10^{0-0}$	$= 10^0$
Divide by 10 =	$\frac{1}{10}$	$= \frac{10^0}{10^1}$	$= 10^{0-1}$	$= 10^{-1}$
Divide by 10 =	________	= ________	= ________	= ________
Divide by 10 =	________	= ________	= ________	= ________

b. Describe what happens to the exponents as the number becomes less.

c. Write each of the powers as a decimal.

6. Rewrite each sequence of numbers using the definition of powers.

a. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, 1, 2, 4, 8

b. $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}$, 1, 3, 9, 27

c. Describe the exponents in the sequence.

7. Write an equivalent expression in simplest form using the quotient of powers rule. Then, write each as a decimal.

a. $\frac{10^0}{10^3}$

b. $\frac{10^0}{10^5}$

c. $\frac{10^0}{10^4}$

8. Complete the table shown.

Unit	Number of Grams	Number of Grams as an Expression with a Positive Exponent	Number of Grams as an Expression with a Negative Exponent
Milligram	$\frac{1}{1000}$		10^{-3}
Microgram		$\frac{1}{10^6}$	
Nanogram	$\frac{1}{1,000,000,000}$		10^{-9}
Picogram		$\frac{1}{10^{12}}$	

9. Write an equivalent expression in simplest form such that the exponent is positive.

a. 8^{-4}

b. 5^{-6}

c. p^{-5}

d. $(4^{-2})(3^{-3})$

10. Complete the table.

Given Expression	Expression with a Positive Exponent	Value of Expression
$\frac{1}{3^{-2}}$		
$\frac{1}{4^{-2}}$		
$\frac{1}{5^{-2}}$		
$\frac{2^{-2}}{1}$	$\frac{1}{2^2}$	$\frac{1}{4}$
$\frac{3^{-2}}{1}$		
$\frac{5^{-2}}{1}$		

11. Describe how to rewrite any expression with a negative exponent in the numerator.

12. Describe how to rewrite any expression with a negative exponent in the denominator.

Talk the Talk

Simplifying

In this lesson, you have developed rules for operating with powers. A summary of these rules is shown in the table.

Properties of Powers	Words	Rule
Product rule of powers	To multiply powers with the same base, keep the base and add the exponents.	$a^m a^n = a^{m+n}$
Power of a power rule	To simplify a power of a power, keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$
Quotient of powers rule	To divide powers with the same base, keep the base and subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$, if $a \neq 0$
Zero power	The zero power of any number, except for 0, is 1.	$a^0 = 1$, if $a \neq 0$
Negative exponents in the numerator	An expression with a negative exponent in the numerator and a 1 in the denominator equals 1 divided by the power with its opposite exponent placed in the denominator.	$a^{-m} = \frac{1}{a^m}$, if $a \neq 0$ and $m > 0$
Negative exponents in the denominator	An expression with a negative exponent in the denominator and a 1 in the numerator equals the power with its opposite exponent.	$\frac{1}{a^{-m}} = a^m$, if $a \neq 0$ and $m > 0$

Write an equivalent expression in simplest form using the properties of powers.

1. $2a^8 \cdot 2a^6$

2. $4b^2 \cdot 8b^9$

3. $-3c \cdot 5c^3 \cdot 2c^9$

4. $(3d^2)^3$

5. $(10ef^3)^5$

6. $\frac{f^8}{f^3}$

7. $\frac{10g^4}{5g^6}$

8. $\frac{30h^8}{15h^2}$

9. $\frac{35i^7j^3}{7i^2j^3}$

10. $\left(\frac{a^2}{a^5}\right)^0$

11. $\frac{2^2}{2^6}$

12. $(4x^2)(3x^5)$

13. $(9^4)(9^{-5})$

14. $(8^0)(8^{-2})$

15. $\frac{3^{-3}}{3^{-3}}$

16. $\frac{4^{-2}}{4^{-3}}$

17. $\frac{(-3)^2}{(-3)^4}$

18. $\frac{h^3}{h^5}$

19. $\frac{x^4}{x^5}$

20. $\frac{m^2p^{-2}}{m^4p^3}$

Lesson 1 Assignment

Write

Use the term *base*, *power*, or *exponent* to complete each sentence.

1. The __________ of a power is the number of times that the factor is repeatedly multiplied.

2. An expression used to represent a factor as repeated multiplication is called a __________.

3. The __________ of a power is the repeated factor in a power.

Remember

Properties of Powers	Words	Rule
Product of powers rule	To multiply powers with the same base, keep the base and add the exponents.	$a^m a^n = a^{m+n}$
Power of a power rule	To simplify a power of a power, keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$
Quotient of powers rule	To divide powers with the same base, keep the base and subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$, if $a \neq 0$

Practice

1. As the principal of a middle school, Mr. Park is in charge of notifying his staff about school delays or cancellations due to weather, power outages, or other unexpected events. Mr. Park starts a phone chain by calling three staff members. Each of these staff members then calls three more staff members, who each call three more staff members. This process completes the calling list.

   a. Excluding Mr. Park, how many staff members are there at the middle school? Explain your calculation.

Lesson 1 Assignment

b. Complete the first three columns in the table.

	Round 1	Round 2	Round 3	Round 4
Number of calls made				
Expanded notation				
Power				

c. Dr. Martinez, superintendent of the school district, decides that she should start the phone chain instead of the school principals. She starts the calling list by calling each of the principals of the three schools in her district. The principals continue the phone chain as described previously. Explain why the table in part (b) can be used to represent Dr. Martinez's phone chain.

d. Complete the fourth column in the table to represent the fourth round of calls in Dr. Martinez's phone chain.

e. How many calls are made in the fourth round?

f. Excluding Dr. Martinez, how many principals and staff members are there in the entire school district? Explain your calculation.

Lesson 1 Assignment

2. The *hertz* (Hz) is a unit of frequency that represents the number of complete cycles per second. It is used to measure repeating events, both scientific and general. For instance, a clock ticks at 1 Hz. One scientific application is the electromagnetic spectrum, or the range of all possible frequencies of electromagnetic waves. The spectrum includes frequencies from everyday contexts, such as radio and TV signals, microwaves, light (infrared, visible, and ultraviolet), and X-rays.

Name	Frequency
1 kilohertz (kHz)	1000 Hz
1 megahertz (MHz)	1000 kHz
1 gigahertz (GHz)	1000 MHz
1 terahertz (THz)	1000 GHz

For each question, use powers to write a mathematical expression. Then, evaluate each expression. Express your answer as a power.

a. How many hertz are in 1 gigahertz? 1 terahertz?

b. A television channel has a frequency of 60 megahertz. What is the channel's frequency in hertz?

c. The frequency of a microwave is 30 gigahertz. What is the microwave's frequency in hertz?

Lesson 1 Assignment

d. The frequency of a visible ray of light is 1000 terahertz. A radio station transmits at a frequency of 100 megahertz. How many times greater is the frequency of the light than the frequency of the radio station?

3. Each expression has been simplified incorrectly. Explain the mistake that occurred and then make the correction.

a. $(-2x)^3 = 8x^3$

b. $\frac{16x^5}{4x} = 12x^4$

c. $(x^2y^4)^3 = x^6y^7$

d. $(x^5y^7)(x^2yz) = x^7y^7z$

4. When you take a picture, the camera shutter controls how much light reaches the film or the digital image sensor. The shutter speed is the amount of time, in seconds, that the shutter stays open. Write each shutter speed as a power with a negative exponent.

a. $\frac{1}{4}$ second

b. $\frac{1}{8}$ second

Lesson 1 Assignment

5. True or False: A number raised to a negative power is always a negative number. Give an example to support your answer.

6. Give an example of a number raised to a negative exponent that is a negative number.

7. Write an equivalent expression in simplest form using the properties of powers. Show your work.

a. $\frac{10^2}{10^5}$

b. $\frac{3^{-5}}{3^{-5}}$

c. $(6x^4)(2x^{-2})$

d. $(7^{-6})(7^4)$

e. $(4^0)(4^{-3})$

f. $\frac{5^2}{5^{-2}}$

g. $\frac{4^{-1}}{4^2}$

h. $\frac{15p^4}{3p^9}$

i. $\frac{m^{-2}}{m^{-6}}$

j. $\frac{q^{-2}\,r^{-3}}{q^6\,r^{-4}}$

Lesson 1 Assignment

Prepare

Simplify each expression.

1. $\frac{1}{3^2}$

2. $-(2^5)^2$

3. $\frac{2}{5} \cdot 10$

4. $\frac{32 + 8}{80 \div 2}$

2 Analyzing Properties of Powers

OBJECTIVES

- Review the product of powers property.
- Review the power of a power property.
- Review the quotient of powers property.
- Generate equivalent expressions by applying properties of integer exponents.

You have learned about the properties of powers with integer exponents.

How can you use these properties to justify your reasoning when solving problems?

Getting Started

Are They Equal?

The symbol for "is equal to" is $=$. The symbol for "is not equal to" is $\neq$. Write the appropriate symbol in each box to compare the two expressions. Explain your reasoning.

1. 2^3 ☐ 2^{-3}

2. $\frac{1}{2^3}$ ☐ 2^{-3}

3. $\frac{1}{2^{-3}}$ ☐ 2^3

4. $\frac{1}{2^{-3}}$ ☐ 2^{-3}

ACTIVITY 2.1 Using Properties of Powers to Justify Steps in Simplifying

Analyze the Worked Example.

WORKED EXAMPLE

$\left(\frac{g^5}{g^4}\right)^3 =$

$= (g^1)^3$ quotient of powers rule

$= g^3$ power of a power rule

1. Identify the rule that justifies each step to simplify the expression.

   a. $2^4 \cdot (-4^1)^3$

   $= 2^4 \cdot (-4^3)$ ____________________

   $= (16)(-64)$ ____________________

   $= -1024$

   b. $\frac{(3 \cdot 5^3)^2}{(2 \cdot 5^4)^2}$ ____________________

   $= \frac{(3^2 \cdot 5^6)}{(2^2 \cdot 5^8)}$ ____________________

   $= \frac{3^2}{(2^2 \cdot 5^2)}$ ____________________

   $= \frac{9}{4 \cdot 25}$ ____________________

   $= \frac{9}{100}$ ____________________

Ask Yourself . . .
Did you justify your mathematical reasoning?

2. Write an equivalent expression in simplest form using the properties of powers. Express your answers using only positive exponents.

a. $\frac{2 \cdot 6^3 \cdot 4^5}{4 \cdot 6^3 \cdot 4^2}$

b. $\frac{(4 \cdot 1^2 \cdot 3^3)^4}{(2 \cdot 1^3 \cdot 3^2)^3}$

c. $\frac{(-3x^2y^4)^8}{(-3x^2y^4)^8}$

d. $\frac{(k^5 \cdot k^5)}{k^6}$

ACTIVITY 2.2

Analyze Errors in Applying Properties of Powers

Determine which student(s) used the properties of powers correctly. Explain why the other expressions are not correct.

1. $\frac{n^7m^4}{n^3m^9}$

   Kayla wrote $n^{10}m^{13}$.

   Adriana wrote $\frac{n^4}{m^5}$.

   Mei wrote n^4m^5.

   Who is correct?

2. $\frac{2(3)^{-4}}{5^{-2}}$

   Kayla wrote $\frac{2(5)^2}{3^4}$.

   Adriana wrote $\frac{5^2}{2(3)^4}$.

   Mei wrote $\frac{2(3)^4}{5^2}$.

   Who is correct?

3. Each expression has been simplified incorrectly. Explain the mistake that occurred, and then make the correction.

a. $(-2 \cdot 7)^3 = 8 \cdot 7^3$

b. $\frac{16 \cdot 7^5}{4 \cdot 7} = 12 \cdot 7^4$

c. $(g^2h^4)^3 = g^6h^7$

d. $(a^5b^7)(a^2b \cdot c) = a^7b^7c$

Talk the Talk

Organize the Properties

1. Create graphic organizers for each rule:

   Product of powers rule

   Quotient of powers rule

   Power of a power rule

   Zero power rule

   Negative exponent rule

   As you are creating your representations, consider the following:

Definition in your own words:	How would you describe this property to a friend?
Facts/Characteristics:	Are there specific characteristics if the numbers are positive or negative? Does this property work the same for variables and numbers?
Examples:	Include examples with variables and different types of numbers (e.g., positive, negative, and fractions).
General Rule:	Use variables. Be mindful of when a variable cannot be zero.

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

PRODUCT OF POWERS RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

QUOTIENT OF POWERS RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

POWER OF A POWER RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

ZERO POWER RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

NEGATIVE EXPONENT RULE

EXAMPLES

GENERAL RULE

Lesson 2 Assignment

Write

In your own words, write the quotient of powers rule, product of powers rule, and the power of a power rule. Use examples to illustrate your descriptions.

Remember

Negative exponents in the numerator can be moved to the denominator and become positive, $a^{-m} = \frac{1}{a^m}$, if $a \neq 0$ and $m > 0$.

The zero power of any number, except for 0, is 1.

Practice

Justify each step to simplify each expression. Choose the properties from the box.

Product of powers rule	Power of a power rule	Negative exponent rule	Quotient of powers rule
Zero power rule	Simplify powers	Identity property of multiplication	Commutative property of multiplication

1. $4x^5 \cdot 6x^2y^6 \cdot xy$

2. $(3a^2b^3)(7ab^5)(a^0b^2)$

3. $(4m^2n^5)^3$

4. $(-3x^7y^3z^0)^5$

Lesson 2 Assignment

5. $\frac{27y^8z^5}{-3y^4z^2}$

6. $\frac{-96m^9n^2}{8m^2n^6}$

7. $-2x^5y^3 \cdot 8x^2y^{-5} \cdot x^{-9}y^2$

8. $\frac{(42m^5n^3)(m^4n^2)}{6m^6n^5}$

Prepare

Use the given explicit formula to generate the first four terms of each geometric sequence.

1. $g_n = 2 \cdot 3^{n-1}$

2. $g_n = 8240 \cdot 1.05^{n-1}$

3. $g_n = 100 \cdot \left(\frac{1}{2}\right)^{n-1}$

4. $g_n = (-2) \cdot 4^{n-1}$

3 Geometric Sequences and Exponential Functions

OBJECTIVES

- Write a geometric sequence as an exponential function in the form $f(x) = ab^x$.
- Identify the constant ratio and y-intercept in different representations of exponential functions.
- Recognize when a relationship is exponential.
- Use algebra to show that, for an exponential function in the form $f(x) = ab^x$, the ratio $\frac{f(x + 1)}{f(x)}$ is constant and equal to b, and the y-intercept is represented by the ordered pair $(0, a)$.

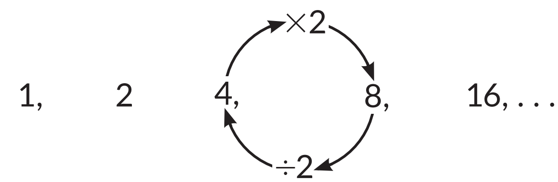
You have learned about geometric sequences and have briefly explored exponential functions.

How can you use geometric sequences to define exponential functions?

Getting Started

Compare and Contrast

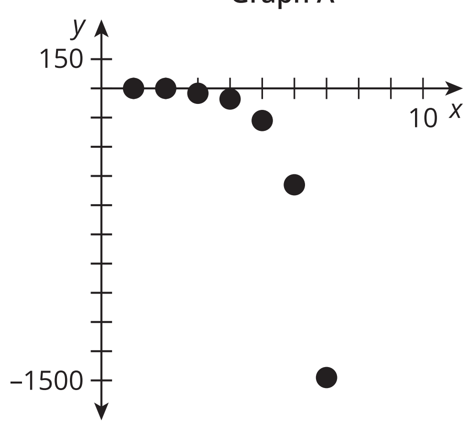
Recall that a *geometric sequence* is a sequence of values in which consecutive terms are separated by a *common ratio*, or constant ratio. For example, the sequence shown is a geometric sequence with a constant ratio of 2.



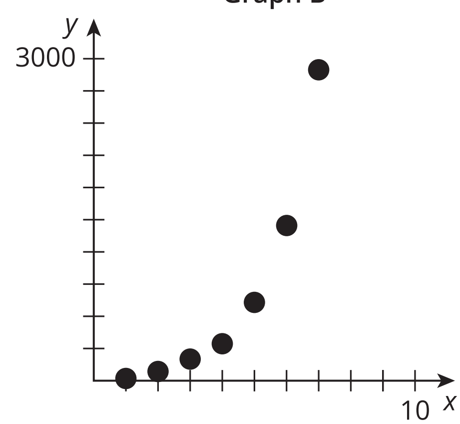
1. Consider a geometric sequence defined by the *recursive formula* $f(n) = \frac{1}{2}(n - 1)$. List the first four terms of the sequence when $f(1) = -5$.

The graphs of six different geometric sequences are shown.

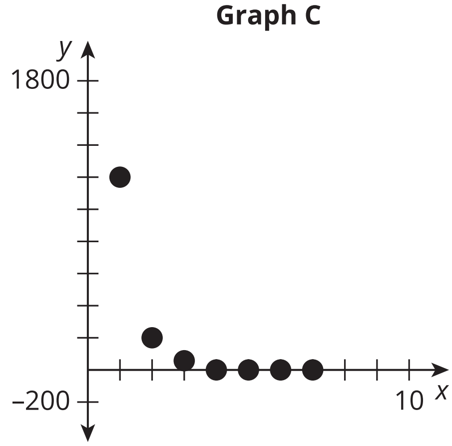
Graph A



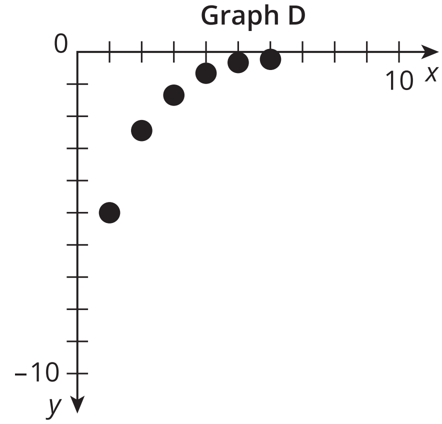
Graph B

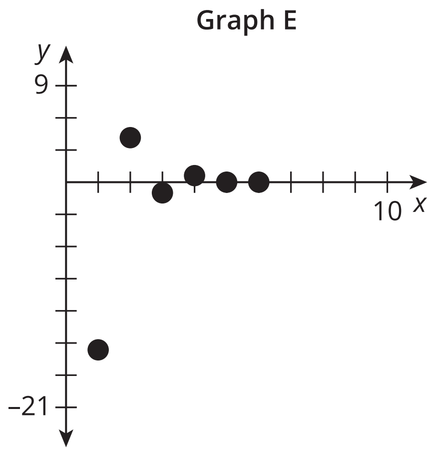


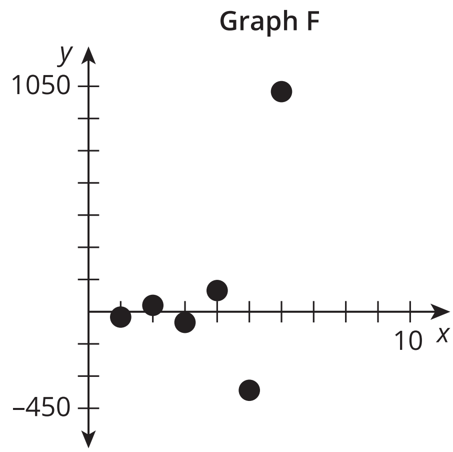
Graph C



Graph D







2. Identify similarities and differences among the graphs. What do you notice?

3. Which of the graphs could represent the sequence given in Question 1? Explain your reasoning.

ACTIVITY 3.1 Identifying the Constant Ratio in Geometric Sequences

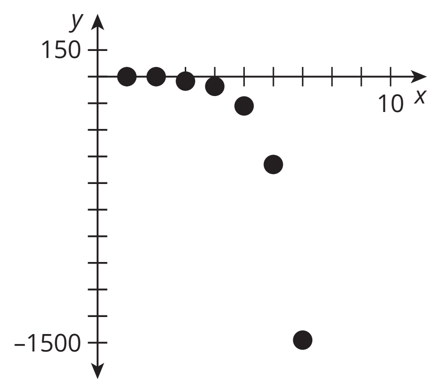
A table of values, a graph, and the explicit formula are given for six geometric sequences.

The *explicit formula* for a geometric sequence is $g_n = g_1(r)^{n-1}$.

1. Identify the constant ratio in each representation.

a. Sequence *A*

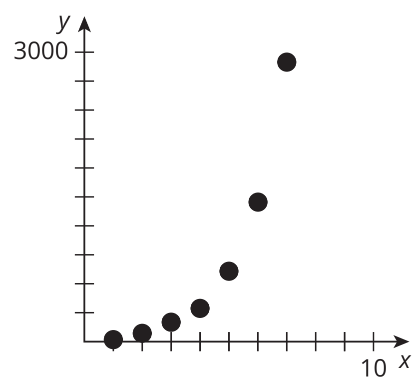
x	y
1	−2
2	−6
3	−18



$y = -2(3)^{x-1}$

b. Sequence *B*

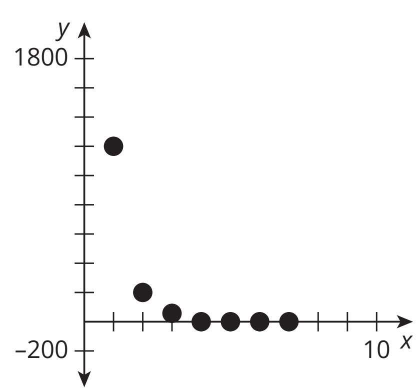
x	y
1	45
2	90
3	180



$y = 45 \cdot 2^{x-1}$

c. Sequence *C*

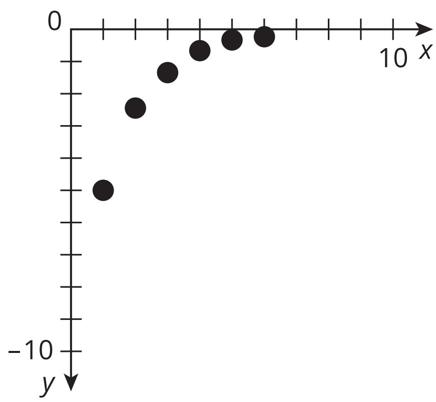
x	y
1	1234
2	123.4
3	12.34



$y = 1234(0.1)^{x-1}$

d. Sequence *D*

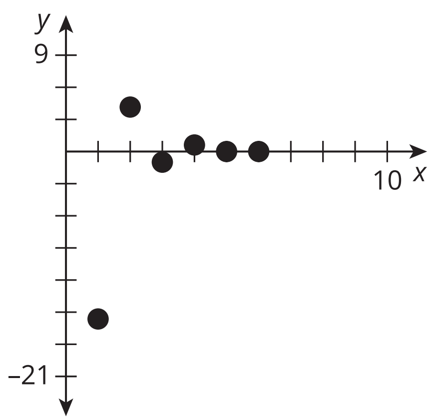
x	y
1	−5
2	−2.5
3	−1.25



$y = -5\left(\frac{1}{2}\right)^{x-1}$

d. Sequence *E*

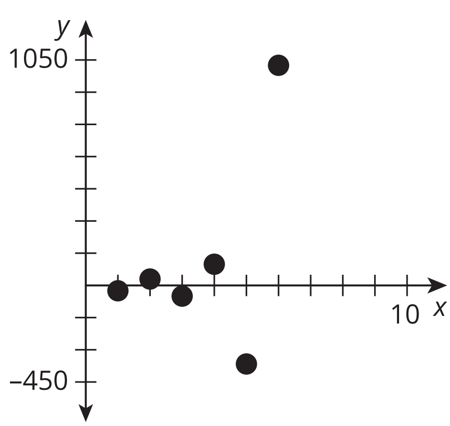
x	y
1	−16
2	4
3	−1



$y = -16\left(-\frac{1}{4}\right)^{x-1}$

f. Sequence *F*

x	y
1	−4
2	12
3	−36



$y = -4(-3)^{x-1}$

2. What strategies did you use to identify the constant ratio for each sequence?

3. Analyze the graphs of the geometric sequences. Do any of the graphs appear to belong to a specific function family? If so, identify the function family. Explain your reasoning.

ACTIVITY 3.2 Exponential Growth

> **Remember . . .**
> All arithmetic sequences can be represented as linear functions. Is there a function family that can represent geometric sequences?

A famous legend tells the story of the inventor of the game of chess. When the inventor showed the new game to the emperor of India, the emperor was so astonished, he said to the inventor, "Name your reward!"

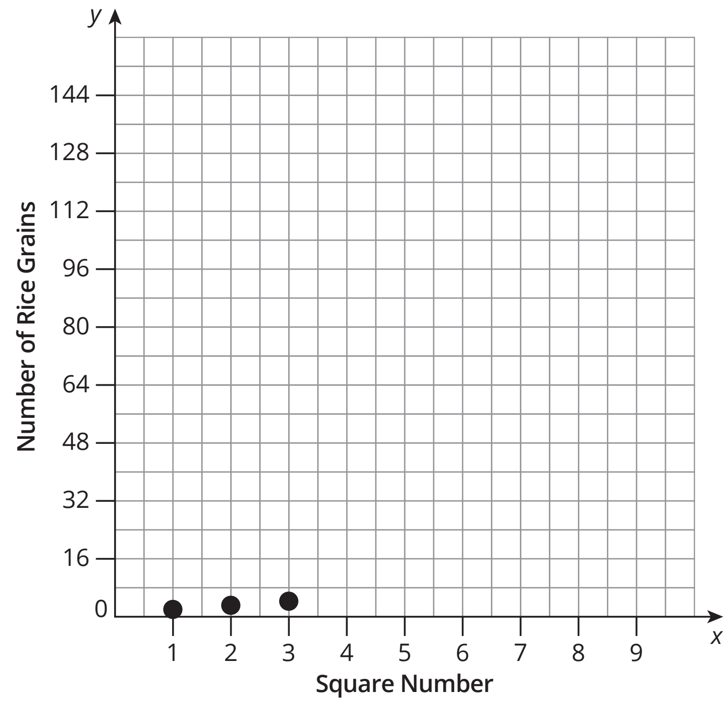
The wise inventor asked the emperor for 1 grain of rice for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, 8 grains for the fourth square, and so on.

Square Number	Number of Rice Grains	Power
1	1	
2	2	
3	4	
4	8	
5		
6		
7		
8		

1. Determine the number of rice grains on the next 4 squares and include them in the table. Complete the third column by writing each number of rice grains as a power with the same base.

2. What pattern do you notice in the table?

3. Graph the points from your table. The first few points have been plotted. Describe the meaning of the plotted points and then identify the function family represented.



4. Identify the constant ratio in the graph, in the table, and in the situation.

5. Angelina and Gabriel each used different methods to write an exponential function to represent the number of rice grains for any square number on the chessboard.

Angelina

I compared the exponents of the power to the square number in the table. Each exponent is 1 less than the square number.

$$f(s) = 2^{s-1}$$

Gabriel

I know the constant ratio is 2. If I extend the pattern back, I get the y-intercept of $(0, \frac{1}{2})$, so I can rewrite the function as

$$f(s) = \frac{1}{2}(2)^s$$

Use properties of exponents to verify that 2^{s-1} and $\frac{1}{2}(2)^s$ are equivalent.

The function that Gabriel wrote is in exponential form. Recall that an exponential function is a function of the form $f(x) = ab^x$, where a and b are real numbers and b is greater than 0, but is not equal to 1.

6. What do the a-value and b-value represent in terms of the equation and graph?

7. Use the exponential function and a calculator to determine the number of rice grains that would be on the very last square of the chessboard. A chessboard has 64 squares.

You can write the explicit formula for geometric sequences in function notation.

Think About . . .

The product of powers rule of exponents allows you to rewrite the product of two powers with the same base:

$(2)^n(2^{-1}) = 2^{n-1}$

$(6)^x(6^y) = 6^{x+y}$

WORKED EXAMPLE

Represent $g_n = 45(2)^{n-1}$ as a function in the form $f(x) = ab^x$.

$g_n = 45 \cdot 2^{n-1}$ or $45(2)^{n-1}$

$f(n) = 45(2)^{n-1}$

Next, rewrite the expression $45(2)^{n-1}$.

$f(n) = 45(2^n)(2^{-1})$	product of powers rule
$f(n) = 45(2^{-1})(2)^n$	commutative property
$f(n) = 45\left(\frac{1}{2}\right)(2)^n$	definition of negative exponent
$f(n) = \frac{45}{2}(2)^n$	multiply.

So, $g_n = 45(2)^{n-1}$ written in function notation is $f(n) = \frac{45}{2}(2)^n$, or $f(n) = 22.5(2)^n$.

In the previous activity, you identified some of the geometric sequences as exponential functions and some that were not exponential functions.

8. Rewrite each explicit formula of the geometric sequences that are exponential functions in function form. Identify the constant ratio and the y-intercept.

Sequence	Explicit Formula	Exponential Function $f(x) = ab^x$	Constant Ratio	y-Intercept
A	$-2(3)^{x-1}$			
B	$45 \cdot 2^{x-1}$			
C	$1234(0.1)^{x-1}$			
D	$-5\left(\frac{1}{2}\right)^{x-1}$			

Based on the graphs of Sequences E and F, you can tell they do not represent exponential functions.

9. Rewrite each explicit formula in function form and explain why these geometric sequences are not exponential functions.

a. Sequence E: $y = -4(-3)^{x-1}$

b. Sequence F: $y = -16\left(-\frac{1}{4}\right)^{x-1}$

10. You know that all arithmetic sequences are linear functions. What can you say about the relationship between geometric sequences and exponential functions?

11. Complete the table by writing each part of the exponential function that corresponds to each part of the geometric sequence.

Geometric Sequence $g_n = g_1 \cdot r^{n-1}$	Exponential Function $f(x) = ab^x$	Mathematical Meaning
g_n		
$\frac{g_1}{r}$		
r		
n		

ACTIVITY
3.3 Identifying Exponential Functions

As part of a project in health class, Valentina, Logan, and Emily are raising awareness and challenging others to eat a healthy breakfast each morning. Today, they each sent selfies of themselves eating a healthy breakfast to 4 friends and challenged them to do the same the next day. This next day, when others send selfies of themselves eating a healthy breakfast will be considered Day 1 of their results. The following day, only those contacted the previous day will send selfies to 4 friends, and the challenge will continue to spread. Assume everyone contacted completes the challenge and new participants are contacted each day.

The constant ratio of an exponential function must be greater than 0 and not equal to 1.

1. Write an exponential function, $f(x)$, to represent the number of new participants of the challenge as a function of the day number, x.

The results after 4 days of the challenge are shown in the table.

Time (Day)	Number of New Participants
1	12
2	48
3	192
4	768

2. The relationship between time and number of participants is exponential.

   a. Verify the relationship is exponential by identifying the constant ratio.

b. What is the number of new participants for Day 0? Explain your answer.

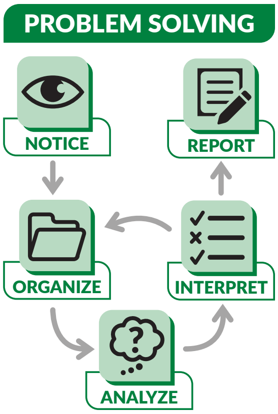
c. If the number of new participants for Day 0 is represented by $f(x) = ab^x$, then the number of new participants for Day 1 can be represented by $f(x + 1) = ab^{(x + 1)}$. Complete the table to show the number of new participants as a function of the day in terms of x and $f(x)$.

Ask Yourself . . .
How can you organize and record your mathematical ideas?

Time (Day)	Number of New Participants		Function Form
x	$f(x)$		$f(x) = ab^x$
0	$f(x)$		$f(x) = ab^x$
1	$f(x + 1)$	12	$f(x + 1) = ab^{(x + 1)}$
2		48	
3		192	
4		768	

d. Use the expressions from the Function Form column of the table and algebra to prove that the table shows a constant ratio between consecutive output values of the function.

Recall that the quotient of powers rule states that when dividing powers with the same base, you can subtract their exponents.
$\frac{b^{(x + 2)}}{b^{(x + 1)}} = b^{(x + 2) - (x + 1)}$

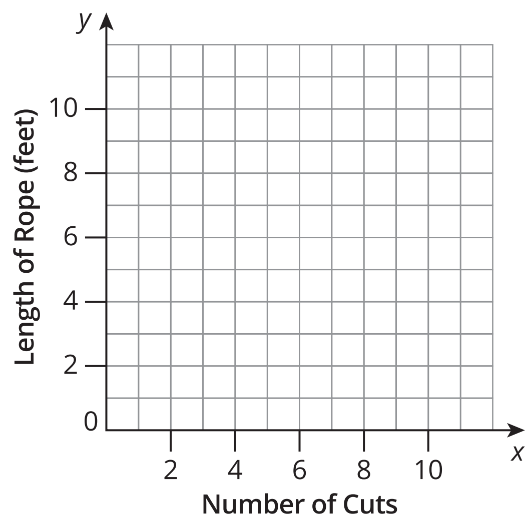


ACTIVITY 3.4 Writing Exponential Functions

A magician is practicing one of his tricks. As part of the trick, he cuts a rope into many pieces and then magically puts the pieces of rope back together. He begins the trick with a 10-foot rope and then cuts it in half. He takes one of the halves and cuts that piece in half. He keeps cutting the pieces in half until he is left with a piece so small he can't cut it anymore.

1. Complete the table to show the length of rope after each of the magician's cuts. Write each length as a whole number, mixed number, or fraction. Then, graph the points from the table.

Number of Cuts	Length of Rope (feet)
0	
1	
2	
3	
4	
5	

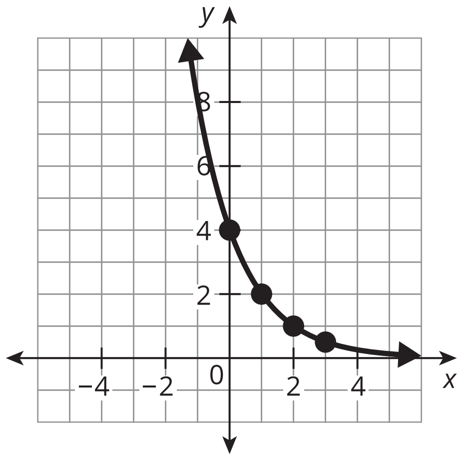


2. Write the function, $L(c)$, to represent the length of the rope as a function of the cut number, c.

3. Use your function to determine the length of the rope after the 7^{th} cut.

4. Write an exponential function of the form $f(x) = ab^x$ for each table and graph, if possible.

a.



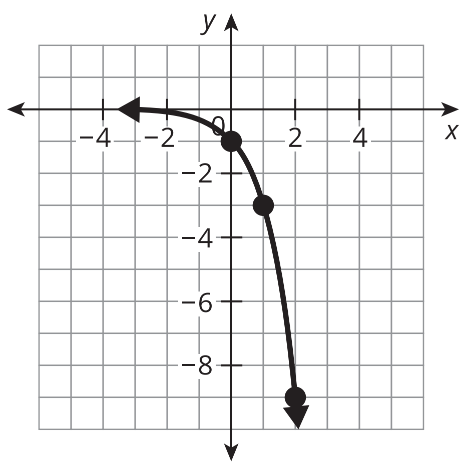
b.

x	y
−2	$-\frac{1}{2}$
−1	−2
0	−8
1	−32

c.

x	y
0	3
1	8
2	13
3	18

d.



Talk the Talk

Did We Mention Constant Ratio?

1. For an exponential function of the form $f(x) = ab^x$, what is the relationship between the base of the power, the expression $\frac{f(x + 1)}{f(x)}$ and the common ratio of the corresponding geometric sequence?

2. How can you decide whether a geometric sequence of the form $g_n = g_1 \cdot b^{n-1}$ represents an exponential function?

Lesson 3 Assignment

Write

Describe the differences between a linear function and an exponential function using your own words.

Remember

All sequences are functions, and some geometric sequences are exponential functions.

The form of an exponential function is $f(x) = ab^x$, where a and b are real numbers and $b > 0$, but $b \neq 1$. The a-value represents the y-intercept, and the b-value represents the constant ratio, or constant multiplier.

Practice

1. Each table shows the population of a city over a three-year period. Write an exponential function to represent each population as a function of time. Round the base to the nearest thousandth.

a.

Blueville	
Time (years)	Population
1	7098
2	7197
3	7298

b.

Yellowville	
Time (years)	Population
1	12,144
2	12,290
3	12,437

c.

Greenville	
Time (years)	Population
1	7860
2	7722
3	7587

2. Consider each situation. If possible, identify a constant ratio and write an exponential function to represent the relationship. Be sure to define your variables.

   a. Diego works in a lab. The number of bacteria over time in a petri dish he is studying is shown in the table.

Bacteria	
Time (years)	Population
0	605
1	2420
2	9680
3	38,720

Lesson 3 Assignment

b. Sarah has been studying the honey bee population. The number of honey bees she documents over time is shown in the table.

Honey Bee Population	
Time (years)	Number of Honey Bees
1	52,910
2	43,069
3	35,058
4	28,537

c. Camilla started depositing money into a savings account. The amount of money over time in the account is shown in the table.

Savings Account	
Time (years)	Value ($)
5	875
10	1200
15	1525
20	1850

Prepare

Write the square root for each perfect square.

1. $\sqrt{16}$

2. $\sqrt{81}$

3. $\sqrt{196}$

4. $\sqrt{225}$

Write the prime factorization of each number.

5. 16

6. 81

7. 196

8. 225

4 Rewriting Square Roots

OBJECTIVES

- Simplify numerical radical expressions involving square roots.
- Multiply and divide with square roots.

You have used operations to compose numeric expressions. How can you operate with numeric radical expressions?

NEW KEY TERMS

- perfect square
- square root
- radical
- radicand
- product property of radicals
- extracting perfect squares
- index
- quotient property of radicals

Getting Started

Building Perfect Squares

Each of these models shows a *perfect square*. A **perfect square** is the product of a whole number multiplied by itself.

1

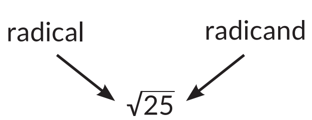
1	1
1	1

1	1	1
1	1	1
1	1	1

Remember . . .

A **square root** is one of two equal factors of a given number. For example, 3 is a square root of 9 because (3)(3) = 9.

The symbol √ is called a **radical**. The **radicand** is the quantity under a radical.



1. Write the area of each perfect square as a power and as a whole number.

2. Write the side length of each perfect square as a radical and as a whole number.

3. Consider the interior squares of each perfect square model.

   a. What is the area of each interior square as a power?

   b. What is the side length of each interior square as a radical?

   c. How could you represent the side length of each perfect square model as a multiplication expression?

4. Can you draw a perfect square model with an area of 20 square units? Why or why not?

Isaiah

Since calculating a square root means to divide by 2, $\sqrt{20} = 10$.

5. Explain why Isaiah is incorrect.

ACTIVITY
4.1 Rewriting Non-Perfect Squares

Consider this square model. Each interior square has an area of 5 square units.

5	5
5	5

1. What is the total area of the square model?

Ask Yourself . . .
Is this a perfect square model?

2. Write the side length of the square model as a radical.

3. Write the side length of the square model as a multiplication expression.

4. Consider the expressions you wrote in Questions 2 and 3. What can you conclude?

5. Write the side length of each square model as a radical and as a multiplication expression.

a.

3	3
3	3

b.

2	2	2
2	2	2
2	2	2

c.

7	7	7
7	7	7
7	7	7

d.

5	5	5	5
5	5	5	5
5	5	5	5
5	5	5	5

e.

4	4	4
4	4	4
4	4	4

f.

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

6. Consider your answers to Question 5. How can you tell when a square model represents a perfect square? Explain your thinking.

Avery noticed a pattern and made a conjecture. She said, "I can write $\sqrt{20}$ as $\sqrt{(4 \cdot 5)}$. Since the model shows that $\sqrt{20}$ is equal to $2 \cdot \sqrt{5}$, that means that $\sqrt{(4 \cdot 5)} = \sqrt{4} \cdot \sqrt{5}$. This will work for all radicals!"

7. Do you think Avery's conjecture is correct? Use the models from Question 5 to explain your reasoning.

Parker

I see a connection to division too!

$$\frac{\sqrt{12}}{\sqrt{4}} = \sqrt{\left(\frac{12}{4}\right)}$$

8. Explain Parker's reasoning.

ACTIVITY

4.2 Multiplying and Dividing Radicals

The variable n in the expression $\sqrt[n]{a}$ is called the **index**. For square roots, the index is 2. The index is often not written for square roots.

Avery and Parker discovered two important properties of radicals.

The **product property of radicals** states that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, when a and b are greater than 0.

Suppose you have the product $\sqrt{15} \cdot \sqrt{5}$. You can use the product property of radicals to rewrite this radical expression.

WORKED EXAMPLE

$$\begin{aligned}\sqrt{15} \cdot \sqrt{5} &= \sqrt{15 \cdot 5} \\ &= \sqrt{3 \cdot 5 \cdot 5} \\ &= \sqrt{3 \cdot 5^2} \\ &= \sqrt{3} \cdot \sqrt{5^2} \\ &= 5\sqrt{3}\end{aligned}$$

Notice that the interior perfect square, 5^2, was extracted from inside the radicand. The process of removing perfect square numbers from under a radical symbol is **extracting perfect squares.**

1. Carlos calculated that $\sqrt{15 \cdot 5} = 3 \cdot 5$, or 15. What error did Carlos make?

2. Rewrite each radical expression by extracting perfect squares.

   a. $\sqrt{50}$

   b. $\sqrt{24}$

   c. $3\sqrt{20}$

   d. $\sqrt{3} \cdot \sqrt{6}$

   e. $\sqrt{3} \cdot \sqrt{12}$

   f. $\sqrt{8} \cdot \sqrt{12}$

3. Determine each product. Then rewrite the product by extracting perfect squares, if possible.

a. $\sqrt{6} \cdot \sqrt{2}$

b. $\sqrt{10} \cdot \sqrt{3}$

c. $\sqrt{14} \cdot \sqrt{2}$

d. $\sqrt{8} \cdot \sqrt{7}$

e. $\sqrt{10} \cdot \sqrt{15}$

The **quotient property of radicals** states that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where $a \geq 0$ and $b > 0$.

4. Rewrite each square root. Extract perfect squares if possible.

a. $\sqrt{\frac{15}{64}}$

b. $\sqrt{\frac{81}{10^2}}$

c. $\sqrt{\frac{48}{25}}$

Jasmine

I can continue to rewrite $\frac{\sqrt{15}}{8}$. Since $\frac{15}{8} = 1.875$, then $\frac{\sqrt{15}}{8} = \sqrt{1.875}$.

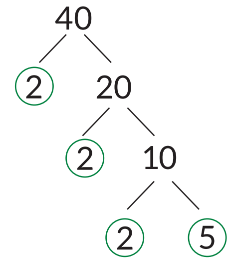
5. Explain how Jasmine reasoned incorrectly.

Another way to rewrite radicals is to use prime factorization.

WORKED EXAMPLE

Rewrite $\sqrt{40}$ in simplest radical form.

Write the prime factorization of 40.



$40 = 2^2 \cdot 2 \cdot 5$

Extract the perfect squares.

$\sqrt{40} = 2 \cdot \sqrt{(2 \cdot 5)}$

Rewrite the radicand.

$\sqrt{40} = 2 \cdot \sqrt{10}$

6. Use prime factorization to rewrite each radical in simplest radical form.

   a. $\sqrt{24}$

   b. $\sqrt{75}$

   c. $\sqrt{35}$

   d. $\sqrt{126}$

7. Consider the different strategies you have learned to rewrite radicals. When might you choose each strategy based on the radicand?

Talk the Talk

Strategy Talk

Consider each square root. Determine the strategy you would use to simplify each radical. Then, rewrite each radical in simplest radical form and show your work.

1. $\sqrt{45}$

2. $\sqrt{81}$

3. $\sqrt{600}$

Lesson 4 Assignment

Write

Explain how you can use the area and side length of a square to understand the components of radical form.

Remember

The product property of radicals states that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$ when a and b are greater than 0. The quotient property of radicals states that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where $a \geq 0$ and $b > 0$.

Rewrite each expression in in simplest radical form.

1. $\sqrt{12}$

2. $\sqrt{30}$

3. $\sqrt{27}$

4. $3\sqrt{75}$

5. $\sqrt{15} \cdot \sqrt{6}$

6. $\sqrt{14} \cdot \sqrt{2}$

7. $\sqrt{8} \cdot \sqrt{7}$

8. $\sqrt{10} \cdot \sqrt{15}$

9. Determine each product. Extract roots if possible.

   a. $\sqrt{5} \cdot \sqrt{20}$

   b. $\sqrt{6} \cdot \sqrt{3}$

   c. $\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{1}{5}}$

   d. $\sqrt{72} \cdot \sqrt{2}$

Lesson 4 Assignment

10. Determine each quotient. Extract roots if possible.

a. $\frac{\sqrt{42}}{\sqrt{7}}$

b. $\frac{\sqrt{26}}{\sqrt{2}}$

c. $\sqrt{\frac{490}{10}}$

d. $\frac{\sqrt{8}}{\sqrt{2}}$

Prepare

Use the properties of exponents to write an equivalent expression in simplest form.

1. $\frac{b^3}{b^0}$

2. $\frac{a \cdot x^5}{a \cdot x^4}$

3. $a \cdot b^2 \cdot a^2 \cdot b^3$

4. $b^{-2} \cdot b^2 \cdot a^0$

5 Rational Exponents and Graphs of Exponential Functions

OBJECTIVES

- Rewrite powers with rational exponents as radical expressions.
- Rewrite radical expressions as powers with rational exponents.
- Use the properties of exponents to interpret output values for non-integer input values in exponential functions.
- Construct exponential functions and identify a common ratio between output values in a graph, a table, and the equation.

NEW KEY TERM

- horizontal asymptote

You have determined the constant ratio of exponential functions with integer inputs.

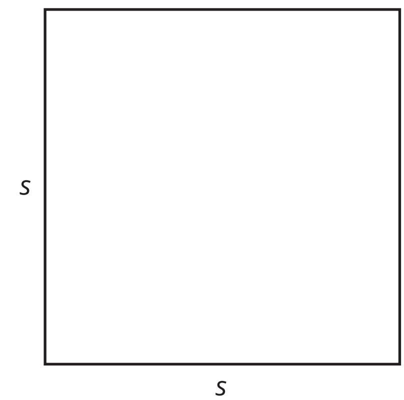
How can you use a constant ratio to determine output values with non-integer inputs?

Getting Started

Squares and Cubes

Write an expression to represent the side length, *s*, of each square given its area. Then, approximate the value of *s*.

1. Area = 5 ft^2

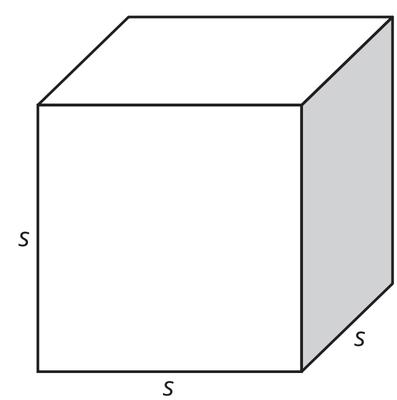


2. Area = 2 cm^2

3. Area = 81 $in.^2$

Write an expression to represent the side length, *s*, of each cube given its volume. Then, approximate the value of *s*.

4. Volume = 51 ft^3



5. Volume = 343 cm^3

6. Volume = 2 $in.^3$

ACTIVITY 5.1

Characteristics of Exponential Growth

In a laboratory experiment, a certain type of bacteria doubles each hour.

1. Suppose a bacteria population starts with just 1 bacterium.

   a. Complete the table to show the population of bacteria, $f(x)$, over time, x.

x	1	2	3	4
$f(x)$				

   b. Determine the constant ratio and y-intercept. Then, write the exponential function that represents the growth of the bacteria population over time. Show your work.

The constant ratio is a multiplier. To determine the next term of a geometric sequence, you multiply by this value.

   c. How is the constant multiplier evident in the problem situation?

2. Graph the exponential function to show bacteria growth over time on the coordinate plane located at the end of the lesson.

ACTIVITY 5.2

The Square Root Constant Ratio of an Exponential Function

The table shown represents the function $f(x) = 2^x$, which models the laboratory experiment that a certain population of bacteria can double each hour.

x	0	1	2	3
$f(x)$	2^0	2^1	2^2	2^3
	1	2	4	8

In the table, the interval between the input values is 1 and the constant multiplier is 2 at the point when the interval changes. What effect, if any, is there on the constant multiplier if the input interval is different?

An exponential function is continuous, meaning that there is a value $f(x)$ for every real number value x.

1. Consider the ratio $\frac{f(2)}{f(0)}$.

   a. Describe the interval of input values. Then, determine the multiplier.

   b. Write two additional ratios that have the same multiplier. Explain your reasoning.

   c. Write a new pair of ratios that have the same multiplier but span a different input interval than the intervals you have already analyzed. Justify your answer.

Nahimana, Lucas, and Luna are interested in the population of bacteria at each $\frac{1}{2}$-hour interval. They have values for the exponential function $f(x)$ when x is an integer. They need the values of the exponential function when x is a rational number between integers.

2. The three students used the idea of the constant multiplier to estimate the value of $f\left(\frac{1}{2}\right)$ for the function $f(x) = 2^x$.

Nahimana

I know the constant multiplier for an interval of 1 is 2. I want to split each interval of 1 into two equal parts, which means I need two equal multipliers.

x	0	$\frac{1}{2}$	1
$f(x)$	2^0	$2^{\frac{1}{2}}$	2^1
	1		2

$1 \cdot r \cdot r = 2$

Multiply by 2.

So, $r^2 = 2$.

Lucas

If r is a constant multiplier for the function as it grows by consecutive integers, it can be split into two equal multipliers of $\sqrt{r}$, because r can be split into two equal factors of $\sqrt{r}$.

$(\sqrt{r})^2 = 2$

Luna

If $f(0) = 2^0 = 1$ and $f(1) = 2^1 = 2$, then $f\left(\frac{1}{2}\right) = 2^{\frac{1}{2}}$ must be equal to 1.5.

a. Use Nahimana's and Lucas's thinking to determine $f\left(\frac{1}{2}\right)$. Write $f\left(\frac{1}{2}\right)$ as a power of 2 and in radical form. Then, enter the values in the table.

x	0	$\frac{1}{2}$	1	2	3
$f(x)$	2^0		2^1	2^2	2^3
	1		2	4	8

b. Use the graph you created in the previous activity to approximate $f\left(\frac{1}{2}\right)$ as a decimal.

c. Explain why Luna's thinking is incorrect.

The number 2 is a rational number because it can be represented as the ratio of two integers. The number $\sqrt{2}$ is an irrational number because it cannot be represented as the ratio of two integers.

Consider the expression $\sqrt{2} = 2^{\frac{1}{2}}$. The square root symbol ($\sqrt{\ }$) is interpreted as the rational exponent $\frac{1}{2}$. All the properties with integer exponents you previously learned continue to apply, even when the exponent is a rational number.

The tables shown represent two different equivalent representations of the constant multiplier, $2^{\frac{1}{2}}$ or $\sqrt{2}$, for the function $f(x) = 2^x$.

	Rational Exponent Representation
$f(0)$	2^0
$f\left(\frac{1}{2}\right)$	$2^{\frac{1}{2}}$
$f(1)$	2^1
$f\left(\frac{3}{2}\right)$	$2^{\frac{3}{2}}$

Arrows between rows: $2^0 \cdot 2^{\frac{1}{2}}$; $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$; $2^1 \cdot 2^{\frac{1}{2}}$

	Radical Form Representation
$f(0)$	1
$f\left(\frac{1}{2}\right)$	$\sqrt{2}$
$f(1)$	2
$f\left(\frac{3}{2}\right)$	$2\sqrt{2}$

Arrows between rows: $1 \cdot \sqrt{2}$; $\sqrt{2} \cdot \sqrt{2}$; $2 \cdot \sqrt{2}$

3. Use the properties of exponents to justify that $2^1 \cdot 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$. Then, use the graph to estimate $2^{\frac{3}{2}}$ as a decimal.

In the expression $\sqrt[n]{a}$, the n is called the *index*.

A rational exponent can be rewritten in radical form using the definition $a^{\frac{1}{n}} = \sqrt[n]{a}$. When the index is 2, it is usually implied rather than written.

Let's consider the properties of exponents to rewrite expressions in equivalent forms.

Remember . . .
The process of removing perfect square numbers from under a radical symbol is called extracting perfect squares.

WORKED EXAMPLE

Consider the expression $2^{\frac{3}{2}}$.

Using the power of a power rule: $2^{\frac{3}{2}} = \left(2^{\frac{1}{2}}\right)^3$ or $(2^3)^{\frac{1}{2}}$.

You can use the definition of rational exponents to rewrite each expression in radical form.

$$\left(2^{\frac{1}{2}}\right)^3 = (\sqrt{2})^3 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 2\sqrt{2}$$

$$(2^3)^{\frac{1}{2}} = \sqrt{2^3} = \sqrt{2 \cdot 2 \cdot 2} = \sqrt{2^2 \cdot 2} = 2\sqrt{2}$$

Remember . . .
The product property of radicals states that $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ when a and b are greater than 0.

You can see that $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = \sqrt{2 \cdot 2 \cdot 2}$.

4. Use the Worked Example to explain the product property of radicals.

Now let's consider how to use the product of powers rule to rewrite the expression with rational exponents in radical form.

WORKED EXAMPLE

Consider the expression $2^{\frac{3}{2}}$.

Using the product of powers rule.

$$
\begin{aligned}
2^{\frac{3}{2}} &= 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} \\
&= \left(2^{\frac{1}{2}}\right)\left(2^{\frac{1}{2}}\right)\left(2^{\frac{1}{2}}\right) \\
&= \sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2} \\
&= \sqrt{2\cdot 2\cdot 2}
\end{aligned}
$$

You can use the definition of *rational exponents* to rewrite each expression in radical form.

5. Analyze the Worked Examples.

   a. Explain why $\sqrt{2}\cdot\sqrt{2} = 2$.

   b. Explain why $\sqrt{2^2} = 2$.

You will learn and practice more with rational exponents later in this lesson.

6. Mason and Sebastian each calculate the population of bacteria when $t = \frac{5}{2}$ hours.

Mason says that when $t = \frac{5}{2}$ hours, $f\left(\frac{5}{2}\right) = \left(\sqrt{2}\right)^5$ bacteria.

Sebastian says that $f\left(\frac{5}{2}\right) = 4\sqrt{2}$ bacteria.

Who's correct?

Use definitions and rules to justify your reasoning. Then, use the graph to estimate the value of $f\left(\frac{5}{2}\right)$ as a decimal on the graph.

ACTIVITY 5.3

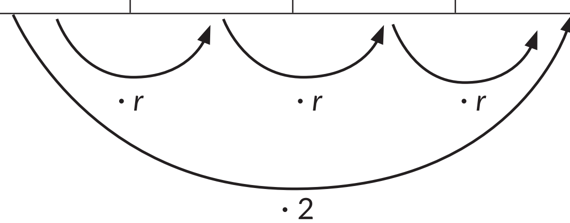
The Cube Root Constant Ratio of an Exponential Function

Think About . . .
What is the constant multiplier you can use to build this relationship over intervals of $\frac{1}{3}$?

In the previous activity, you looked at the exponential function $f(x) = 2^x$. When the input interval is 1, the constant ratio is 2^1, and when the input interval is $\frac{1}{2}$, the constant multiplier is $2^{\frac{1}{2}}$.

Now, let's think about the constant multiplier when the input interval is $\frac{1}{3}$.

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$
$f(x)$	2^0			2^1	
	1			2	



In the expression $\sqrt[n]{a}$, only $n = 2$ is implied rather than written. All other index values must be written.

1. Complete the table of values for the exponential function $f(x) = 2^x$. Represent $f(x)$ as a rational exponent and in radical form. Show your work.

2. Write the points represented in the table as ordered pairs. Use the graph you created in the previous activity to estimate each output value as a decimal.

ACTIVITY 5.4

Negative Exponents and Asymptotes

You have explored output values with integer and rational exponents for the exponential function $f(x) = 2^x$. What happens when x is a negative value?

Remember . . .

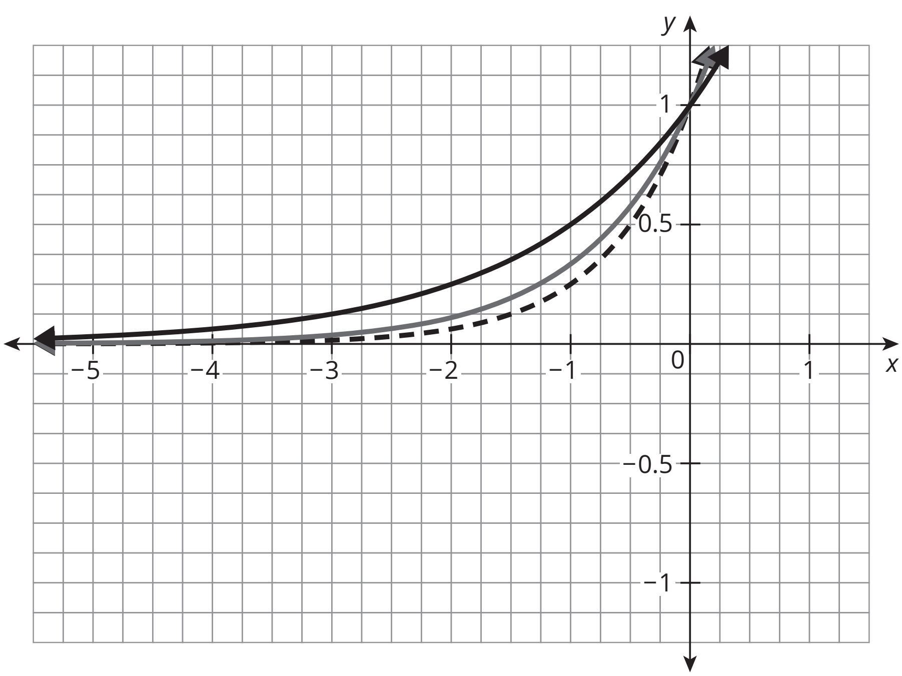
The negative exponent rule states that $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, and so on.

1. Use what you know about negative exponents to complete the table for the function $f(x) = 2^x$.

x	-4	-3	-2	-1	0
$f(x)$					2^0
					1

2. Consider the table of values you just completed. How does $f(x)$ change as x approaches negative infinity?

An exponential function has a *horizontal asymptote*. A **horizontal asymptote** is a horizontal line that a function gets closer and closer to, but never intersects.



3. Label the function $f(x) = 2^x$ on the coordinate plane. Identify the horizontal asymptote.

4. The functions $g(x) = 3^x$ and $h(x) = 4^x$ are also shown on the coordinate plane.

   a. Identify each function and explain how you know.

   b. Determine the horizontal asymptotes for the functions. Compare these with the horizontal asymptote of $f(x) = 2^x$.

5. Compare how each of the three functions approaches its horizontal asymptote. What is the same and what is different?

6. Identify the domain and range of each of the three functions. What do you notice?

ACTIVITY 5.5

Rational Exponents

In this lesson, you have been writing powers with rational exponents. You have shown that you can rewrite a rational exponent in radical form, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

1. Rewrite each expression as a power.

   a. $\sqrt[3]{7}$

   b. $\sqrt[5]{x}$

   c. $\sqrt{y}$

Remember . . .

In the expression $\sqrt[n]{a}$, the n is called the *index*.

2. Rewrite each expression in radical form.

   a. $8^{\frac{1}{4}}$

   b. $z^{\frac{1}{5}}$

   c. $m^{\frac{1}{3}}$

3. Use the properties of exponents to rewrite $a^{\frac{m}{n}}$ in radical form.

4. Rewrite each expression in radical form. Simplify if possible.

a. $4^{\frac{3}{2}}$

b. $5^{\frac{3}{4}}$

c. $x^{\frac{4}{5}}$

d. $4^{\frac{2}{3}}$

Ask Yourself . . .
How does representing mathematics in multiple ways help to communicate reasoning?

5. Rewrite each expression as a power with a rational exponent.

a. $(\sqrt[4]{2})^3$

b. $(\sqrt{5})^4$

c. $\sqrt[5]{x^8}$

d. $(\sqrt[5]{y})^{10}$

Let's analyze the product of radicals.

WORKED EXAMPLE

You can rewrite the numeric expression $(\sqrt[3]{2})^2(\sqrt{2})$ in radical form using the rules of exponents.

$(2^{\frac{1}{3}})^2\,(2)^{\frac{1}{2}}$	Definition of rational exponents
$(2^{\frac{2}{3}})\,(2^{\frac{1}{2}})$	Power of a power rule
$2^{\frac{2}{3}+\frac{1}{2}}$	Product of powers rule
$2^{\frac{7}{6}}$	Add fractions.
$\sqrt[6]{2^7}$	Definition of rational exponents

6. Kayla rewrote the expression $\sqrt[6]{2^7}$ in a different way.

$$2^{\frac{7}{6}} = 2^{\frac{6}{6}+\frac{1}{6}} = 2^{\frac{6}{6}} \cdot 2^{\frac{1}{6}} = 2\sqrt[6]{2}$$

Is she correct? Justify your reasoning.

Let's revisit the product property radical and the process of extracting roots.

Suppose you have the product $\sqrt{15} \cdot \sqrt{5}$. You can use properties of exponents to rewrite this radical expression.

WORKED EXAMPLE

$$\begin{aligned} \sqrt{15} \cdot \sqrt{5} &= 15^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \\ &= (15 \cdot 5)^{\frac{1}{2}} \\ &= (3 \cdot 5 \cdot 5)^{\frac{1}{2}} \\ &= (3 \cdot 5^2)^{\frac{1}{2}} \\ &= 3^{\frac{1}{2}} \cdot 5 \\ &= 5\sqrt{3} \end{aligned}$$

$$\begin{aligned} \sqrt{15} \cdot \sqrt{5} &= \sqrt{15 \cdot 5} \\ &= \sqrt{3 \cdot 5 \cdot 5} \\ &= \sqrt{3 \cdot 5^2} \\ &= 5\sqrt{3} \end{aligned}$$

7. Use either strategy in the Worked Example to rewrite each radical expression by extracting perfect squares.

a. $\sqrt{50}$

b. $\sqrt{24}$

c. $3\sqrt{20}$

d. $\sqrt{3} \cdot \sqrt{6}$

e. $\sqrt{3} \cdot \sqrt{12}$

f. $\sqrt{8} \cdot \sqrt{12}$

8. Explain how the properties of rational exponents extend from the properties of integer exponents.

9. Consider the calculations you made throughout this lesson and the definition of a rational number to answer each question.

   a. Is the product of a nonzero rational number and an irrational number always, sometimes, or never a rational number? Explain your reasoning.

   b. Is the product of an irrational number and an irrational number always, sometimes, or never a rational number? Explain your reasoning.

WORKED EXAMPLE

You can rewrite the numeric expression $\frac{\sqrt[3]{x}\sqrt{x}}{\sqrt[6]{x}}$ in radical form using rules of exponents.

$\frac{x^{\frac{1}{3}}x^{\frac{1}{2}}}{x^{\frac{1}{6}}}$	Rewrite using rational exponents.
$\frac{x^{\frac{1}{3}+\frac{1}{2}}}{x^{\frac{1}{6}}}$	Apply the product of powers rule.
$\frac{x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$	Add fractions.
$x^{\frac{4}{6}}$	Apply the quotient of powers rule.
$x^{\frac{2}{3}}$	Rewrite fraction.
$\sqrt[3]{x^2}$	Rewrite in radical form.

10. Rewrite each expression using the rules of exponents. Write responses in radical form.

a. $\left(3^{\frac{3}{2}}\right)^3$

b. $25^{\frac{3}{2}}$

c. $\left(2x^{\frac{1}{2}}y^{\frac{1}{3}}\right)\left(3x^{\frac{1}{2}}y\right)$

d. $\left(\frac{24m^{\frac{3}{4}}n^{\frac{5}{2}}}{36m^{\frac{2}{7}}n^{\frac{2}{5}}}\right)^0$

Talk the Talk

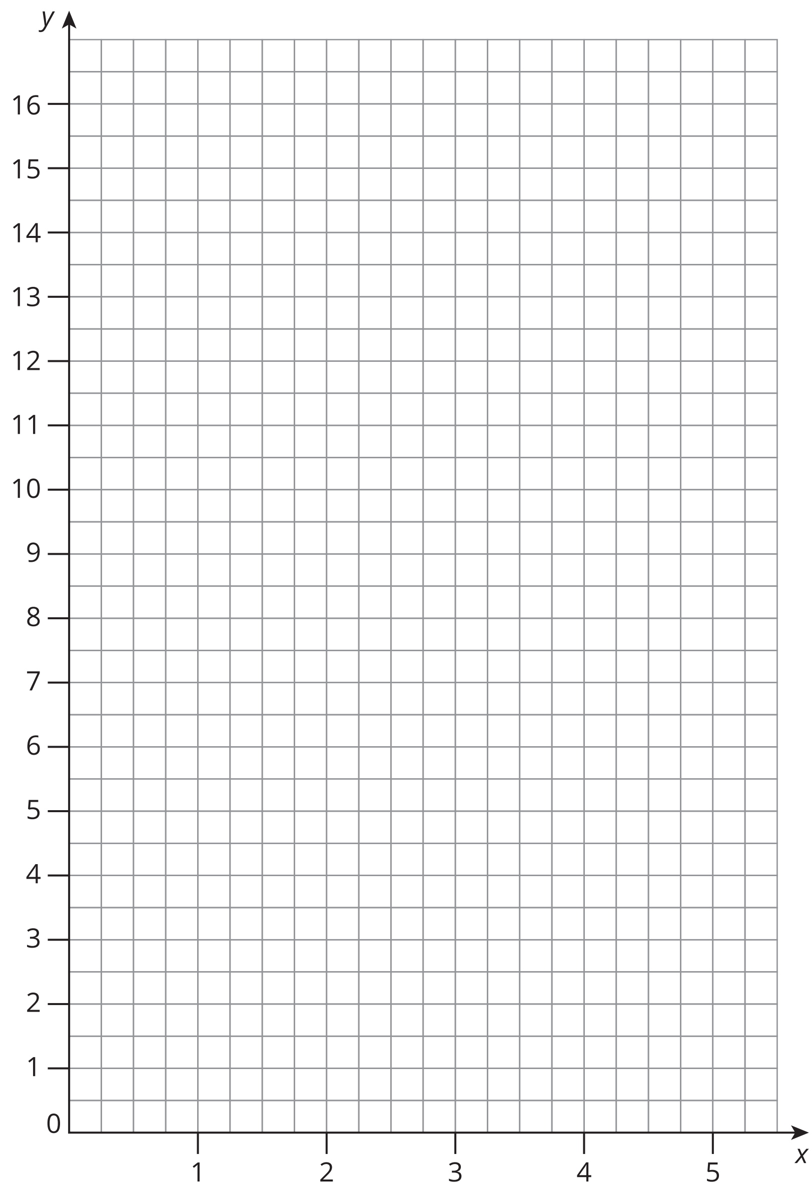
May the Fourths Be With You

1. Consider the exponential function $f(x) = 2^x$. Complete the table. Then, compare the table of fourths to the tables you completed for halves and thirds. What patterns do you notice in the multiplier?

x	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
$f(x)$	2^0				2^1
	1				2

2. Match each rational expression with the appropriate radical expression.

Rational Expression	Radical Expression
1. $2^{\frac{5}{10}}$	A. $\sqrt[5]{10^2}$
2. $10^{\frac{5}{2}}$	B. $\sqrt[5]{2^{10}}$
3. $10^{\frac{2}{5}}$	C. $\sqrt[10]{2^5}$
4. $5^{\frac{10}{2}}$	D. $\sqrt{10^5}$
5. $2^{\frac{10}{5}}$	E. $\sqrt{5^{10}}$
6. $5^{\frac{2}{10}}$	F. $\sqrt[10]{5^2}$


y
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
x

Lesson 5 Assignment

Write

Describe how the components of radical form and rational exponent form of an equivalent expression are related.

Remember

If the difference in the input values is the same, an exponential function shows a constant multiplier between output values, no matter how large or how small the gap between input values.
If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

Practice

Rewrite each radical using a rational exponent.

1. $\sqrt[4]{88}$
2. $\sqrt[10]{46}$
3. $\sqrt[6]{x}$
4. $\sqrt{z}$

Rewrite each power in radical form.

5. $9^{\frac{1}{3}}$
6. $5^{\frac{1}{2}}$
7. $20^{\frac{1}{5}}$
8. $41^{\frac{1}{8}}$

Rewrite each power in radical form. Simplify your answer, if possible.

9. $16^{\frac{3}{2}}$
10. $5^{\frac{7}{4}}$
11. $12^{\frac{2}{5}}$
12. $8^{\frac{4}{3}}$

Lesson 5 Assignment

13. $\left(3x^{\frac{1}{3}}y\right)\left(4x^{\frac{2}{3}}y^{\frac{1}{2}}\right)$

14. $(225x^2y^0z^3)^{\frac{1}{2}}$

Rewrite each expression using a rational exponent. Simplify your answer, if possible.

15. $\sqrt[5]{10^4}$

16. $(\sqrt[4]{t})^4$

17. $(\sqrt{w})^6$

18. $\sqrt[9]{h^3}$

Prepare

Rewrite each expression by combining like terms.

1. $-3x + 4y - 9x - 5y$

2. $2xy^2 + 5x^2y - 7xy + xy^2$

3. $6 - m^2 + 5m^2$

4. $-8 - (-4k) + 7 + 1 - 4k$

TOPIC 1 SELF-REFLECTION

Name: ____________________

Introduction to Exponential Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 **represents I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Introduction to Exponential Functions* topic by:

TOPIC 1: *Introduction to Exponential Functions*	Beginning of Topic	Middle of Topic	End of Topic
using the rules of integer exponents to simplify numeric and algebraic expressions.	◯	◯	◯
justifying the exponent rules, including product of powers, quotient of powers, power of a power, and zero and negative exponent rules.	◯	◯	◯
writing an equation in function notation for an exponential function represented as a table, a graph, a set of ordered pairs, or a scenario.	◯	◯	◯
evaluating an exponential function for any unknown value.	◯	◯	◯
using the equation, graph, or scenario to identify key characteristics of an exponential function, including the y-intercept, the horizontal asymptote, the x-intercept, and intervals of increase or decrease.	◯	◯	◯
describing what the key characteristics of a function indicate about the problem scenario.	◯	◯	◯
creating a graph, including all of the key characteristics, that matches a given scenario.	◯	◯	◯

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Introduction to Exponential Functions*	Beginning of Topic	Middle of Topic	End of Topic
explaining that the parent function for an exponential relationship is $f(x) \cdot ab^x$, where $a \cdot 1$ and $b > 0$.			
demonstrating that an exponential function has a constant ratio over equal intervals.			
identifying an exponential function from a table of values, an equation, a graph, or a scenario.			
simplifying square roots by extracting perfect squares.			
using the rules of exponents to rewrite numeric and algebraic expressions with rational exponents.			
applying the rule $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ to rewrite expressions.			

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Introduction to Exponential Functions* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Introduction to Exponential Functions Summary

LESSON 1 Properties of Powers with Integer Exponents

An expression used to represent the product of a repeated multiplication is a *power*. A **power** has a base and an *exponent*. The **base** of a power is the expression that is used as a factor in the repeated multiplication. The **exponent** of a power is the number of times that the base is used as a factor in the repeated multiplication.

You can write a power as a product by writing out the repeated multiplication.

$$2^7 = (2)(2)(2)(2)(2)(2)(2)$$

The power 2^7 can be read as:

- “two to the seventh power.”
- “the seventh power of two.”
- “two raised to the seventh power.”

Parentheses can change the value of expressions containing exponents. When the negative sign is not in parentheses, it’s not part of the base. For example, $-1^2 = -1$, but $(-1)^2 = 1$.

NEW KEY TERMS

- power [potencia]
- base [base]
- exponent [exponente]
- horizontal asymptote [asíntota horizontal]
- perfect square
- square root
- radical [radical]
- radicand [radicando]
- product property of radicals [propiedad del producto de radicales]
- extracting perfect squares
- index [índice]
- quotient property of radicals [propiedad del cociente de radicales]

Properties of Powers	Words	Rule
Product of powers rule	To multiply powers with the same base, keep the base and add the exponents.	$a^m a^n = a^{m+n}$
Power of a power rule	To simplify a power of a power keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$
Quotient of powers rule	To divide powers with the same base, keep the base and subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$, if $a \neq 0$
Zero power	The zero power of any number, except for 0, is 1.	$a^0 = 1$, if $a \neq 0$
Negative exponents in the numerator	An expression with a negative exponent in the numerator and a 1 in the denominator equals 1 divided by the power with its opposite exponent placed in the denominator.	$a^{-m} = \frac{1}{a^m}$, if $a \neq 0$ and $m > 0$
Negative exponents in the denominator	An expression with a negative exponent in the denominator and a 1 in the numerator equals the power with its opposite exponent.	$\frac{1}{a^{-m}} = a^m$, if $a \neq 0$ and $m > 0$

LESSON 2

Analyzing Properties of Powers

The properties of powers can be used to simplify numeric and algebraic expressions.

For example, you can simplify the expression $\left(\frac{x^5}{x^4}\right)^3$.

$\left(\frac{x^5}{x^4}\right)^3 = (x^1)^3$ quotient of powers rule

$= x^3$ power of a power rule

LESSON 3 Geometric Sequences and Exponential Functions

An exponential function is a function of the form $f(x) = ab^x$, where a and b are real numbers and b is greater than 0 but not equal to 1.

Geometric sequences with positive common ratios belong in the exponential function family. The common ratio of a geometric sequence is the base of an exponential function.

If a geometric sequence represents an exponential function, you can use the product of powers rule and the definition of negative exponents to rewrite the explicit formula for the sequence as an exponential function.

For example, to represent $g_n = 45(2)^{n-1}$ using function notation, first rewrite it as $f(n) = 45(2)^{n-1}$.

Next, rewrite the expression $45(2)^{n-1}$.

$f(n) = 45(2)^n(2)^{-1}$ product of powers rule

$f(n) = 45(2)^{-1}(2)^n$ commutative property

$f(n) = (45)(\frac{1}{2})(2)^n$ definition of negative exponent

$f(n) = \frac{45}{2}(2)^n$ multiply

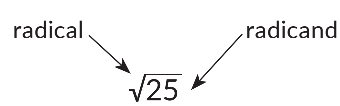
So, $g_n = 45(2)^{n-1}$ written in function notation is $f(n) = \frac{45}{2}(2)^n$.

The variable a in $f(x) = ab^x$ is the y-intercept, and b is the constant ratio.

LESSON 4 Rewriting Square Roots

A **square root** is one of two equal factors of a given number. Every positive number has two square roots: a positive square root and a negative square root. The positive square root is called the *principal square root*.

The symbol $\sqrt{10}$ is called a **radical**. The **radicand** is the quantity under a radical. For example, the expression shown is read as "the square root of 25" or as "radical 25."



A **perfect square** is a number that is equal to the product of an integer multiplied by itself. In the example above, 25 is a perfect square because it is equal to the product of 5 multiplied by itself.

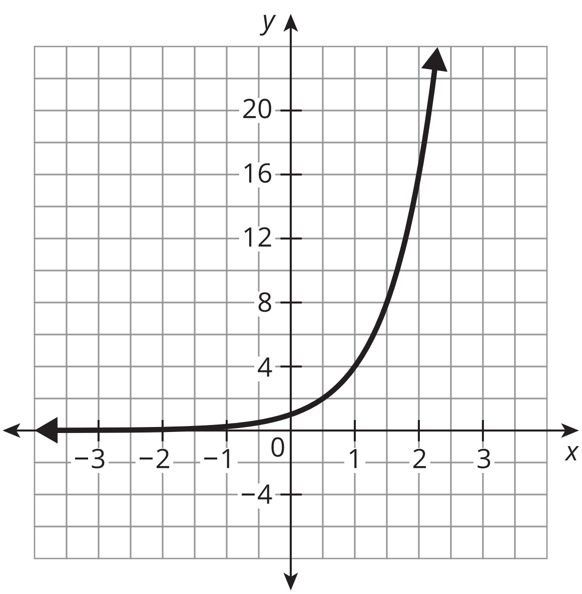
Rewrite radical expressions with an index of 2 by **extracting perfect squares.** This is the process of removing perfect squares from under the radical symbol.

For example, consider the irrational number represented by the radical expression $\sqrt{40}$.

$$\sqrt{40} = \sqrt{(4 \cdot 10)}$$
$$= \sqrt{4} \cdot \sqrt{10}$$
$$= 2\sqrt{10}$$

LESSON 5 Rational Exponents and Graphs of Exponential Functions

An exponential function is continuous, meaning that there is a value $f(x)$ for every real number value x. If the difference in the input values is the same, an exponential function shows a constant ratio between output values, no matter how large or how small the gap between input values. A constant ratio can be used to determine output values for integer and for non-integer inputs.



An exponential function has a **horizontal asymptote**, which is a horizontal line that a function gets closer and closer to but never intersects.

Consider the table and graph represented by the function $f(x) = 4^x$.

There are no x-intercepts. The y-intercept is at (0, 1). The horizontal asymptote is $y = 0$. The domain is all real numbers, and the graph increases over the entire domain. The range is $y > 0$.

A rational exponent is an exponent that is a rational number. You can write each nth root using a rational exponent. If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

For example, $\sqrt[4]{b} = b^{\frac{1}{4}}$ and $6^{\frac{1}{5}} = \sqrt[5]{6}$.

x	$f(x)$
−2	$\frac{1}{16}$
−1	$\frac{1}{4}$
0	1
1	4
2	16

Write expressions with rational exponents in radical form using the known properties of integer exponents. Write the power as a product using a unit fraction. Use the power of a power rule and the definition of a rational exponent to write the power as a radical.

For example, consider the expressions $8^{\frac{2}{3}}$ and $(\sqrt[7]{c})^3$.

$$8^{\frac{2}{3}} = 8^{\left(\frac{1}{3}\right)^2}$$
$$= (\sqrt[3]{8})^2$$

$$(\sqrt[7]{c})^3 = c^{\left(\frac{1}{7}\right)^3}$$
$$= c^{\frac{3}{7}}$$

ALGEBRA I

TOPIC 2

Exponential functions become steeper and steeper or flatter and flatter. The view looking straight down from the top of the Innsbruck Ski Jump is shown. The Winter Olympics were held in Innsbruck, Austria in 1964 and 1976. The ski jump comes down steeply and flattens out, rather like an exponential function. When the ski jump is covered with snow, a jumper can reach speeds of around 55 miles per hour within four seconds of launching off the ramp.

Using Exponential Equations

LESSON 1 Exponential Equations for Growth and Decay **609**

LESSON 2 Interpreting Parameters in Context **623**

LESSON 3 Modeling Using Exponential Functions **637**

1 Exponential Equations for Growth and Decay

OBJECTIVES

- Classify exponential functions as increasing or decreasing.
- Compare formulas for simple interest and compound interest situations.
- Compare the average rate of change between common intervals of a linear and an exponential relationship.
- Write an exponential function that includes a percent increase or decrease with a *b*-value that is a decimal number.
- Solve exponential equations using graphs.

NEW KEY TERMS

- simple interest
- compound interest
- exponential growth function
- exponential decay function

You have analyzed linear and exponential functions and their graphs.

How can you compare linear and exponential functions as increasing and decreasing functions?

Getting Started

Up or Down?

Ask Yourself . . .
What does the structure of each function equation tell you?

Consider each function shown.

$f(x) = -2x + 5$	$g(x) = 2^x$	$h(x) = 0.95^x$
$p(x) = 6\left(\frac{5}{8}\right)^x$	$q(x) = 3(x - 4) - 1$	$r(x) = 2(1 - 0.5)^x$
$v(x) = 4 \cdot 1.10^{(x + 5)}$	$w(x) = -5 \cdot 3^x$	$z(x) = -x + 10$

1. Sort the functions into two groups. Justify your choices.

<u>Increasing Functions</u>   <u>Decreasing Functions</u>

ACTIVITY 1.1 Simple and Compound Interest

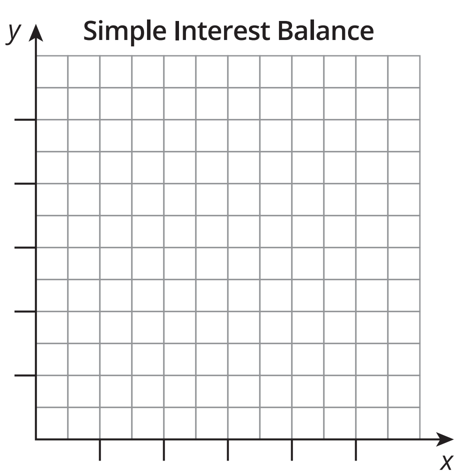
Suppose that your family deposited $10,000 in an interest-bearing account for your college fund that earns simple interest each year. A friend's family deposited $10,000 in an interest-bearing account for their child's college fund that earns compound interest each year.

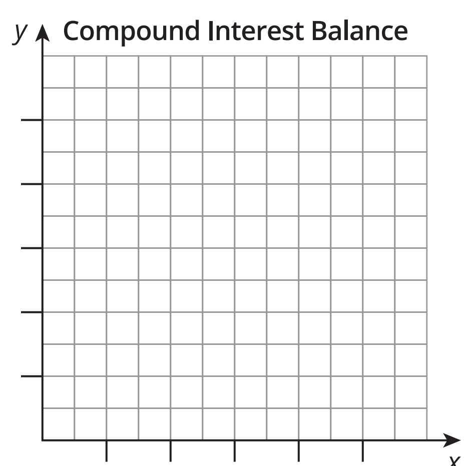
Time (years)	Simple Interest Balance (dollars)	Compound Interest Balance (dollars)
0	10,000	10,000
1	10,400	10,400
2	10,800	10,816
3	11,200	11,248.64
10	14,000	14,802.44

In a **simple interest** account, a percent of the starting balance is added to the account at each interval. The formula for simple interest is $I = Prt$, where P represents the starting amount, or principal, r represents the interest rate, t represents time, and I represents the interest earned. In a **compound interest** account, the balance is multiplied by the same amount at each interval.

1. Study the table of values.

   a. Sketch a graph of each account balance in dollars as a function of the time in years.





   b. Write a function, $s(x)$, to represent the simple interest account and a function, $c(x)$, to represent the compound interest account.

Think about . . .
How accurate does your answer need to be?

2. Use the functions $s(x)$ and $c(x)$ to determine each value.

   a. $s(5)$

   b. $c(5)$

   c. $s(4)$

   d. $c(4)$

3. Determine the average rate of change between each pair of values given for each relationship.

Time Intervals (years)	Simple Interest Function (dollars)	Compound Interest Function (dollars)
Between $t = 0$ and $t = 1$		
Between $t = 1$ and $t = 2$		
Between $t = 2$ and $t = 5$		
Between $t = 5$ and $t = 10$		

4. Compare the average rates of change for the simple and compound interest accounts.

   a. What do you notice?

   b. What does this tell you about the graphs of linear and exponential functions?

5. Use technology to determine when each account will reach the given dollar amount.

   a. When does the simple interest account reach $15,600?

   b. Approximately when does the compound interest account reach $100,000?

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

6. Mei says that given any increasing linear function and any exponential growth function, the output of the exponential function will eventually be greater than the output of the linear function. Is Mei correct? Use examples to justify your thinking.

ACTIVITY 1.2

Identifying Exponential Growth and Decay

Remember . . .

Percent means out of 100. You can write 1.5% as $\frac{1.5}{100}$, which is equivalent to 0.015.

At this moment, the population of Downtown is 20,000, and the population of Uptown is 6000. But, over many years, people have been moving away from Downtown at a rate of 1.5% every year. At the same time, Uptown's population has been growing at a rate of 1.8% each year.

1. What are the independent and dependent quantities in each situation?

2. Which city's population can be represented as an increasing function, and which can be represented as a decreasing function?

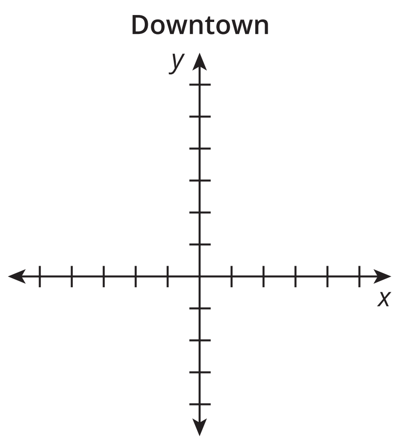
Remember . . .

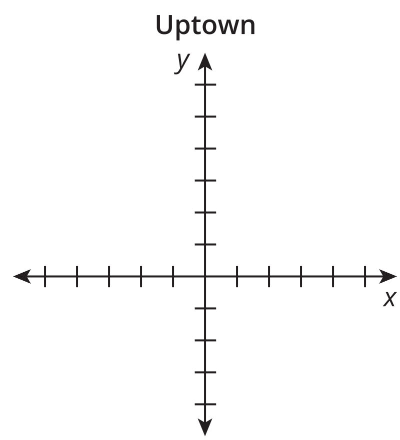
You can use the y-intercept and the horizontal asymptote to help graph an exponential function.

Let's examine the properties of the graphs of the functions for Downtown and Uptown.

Downtown: $D(t) = 20{,}000(1 - 0.015)^t$ Uptown: $U(t) = 6000(1 + 0.018)^t$

3. Sketch a graph of each function. Label key points.





4. The functions $D(t)$ and $U(t)$ can each be written as an exponential function of the form $f(x) = ab^x$.

   a. What is the a-value for each function? What does each a-value mean in terms of this problem situation?

   b. What is the b-value for each function? What does each b-value mean in terms of this problem situation?

   c. Compare and explain the meanings of the expressions $(1 - 0.015)^t$ and $(1 + 0.018)^t$ in terms of this problem situation.

5. Analyze the y-intercepts of each function.

   a. Identify the y-intercepts.

   b. Interpret the meaning of each y-intercept in terms of the problem situation.

   c. Describe how you can determine the y-intercept of each function using only the formula for population increase or decrease.

Think About . . .

A decreasing exponential function is denoted by a decimal or fractional b-value between 0 and 1, not by a negative b-value.

An **exponential growth function** has a b-value greater than 1 and is of the form $y = a(1 + r)^x$, where r is the rate of growth. The b-value is $1 + r$. An **exponential decay function** has a b-value greater than 0 and less than 1 and is of the form $y = a(1 - r)^x$, where r is the rate of decay. The b-value is $1 - r$.

ACTIVITY
1.3

Comparing Exponential Functions

Functions

$f(x) = 7000 \cdot 0.969^x$

$f(x) = 7000(1 + 0.028)^x$

$f(x) = 7000 \cdot 1.012^x$

$f(x) = 7000(1 - 0.0175)^x$

$f(x) = 7000 \cdot 1.014^x$

$f(x) = 7000 \cdot 0.9875^x$

Consider the six different city population scenarios.

1. Match each situation with the appropriate function. Explain your reasoning.

   a. A city has a population of 7000. Its population is increasing at a rate of 1.4%.

   b. A city has a population of 7000. Its population is decreasing at a rate of 1.75%.

   c. A city has a population of 7000. Its population is increasing at a rate of 1.2%.

   d. A city has a population of 7000. Its population is decreasing at a rate of 3.1%.

   e. A city has a population of 7000. Its population is increasing at a rate of 2.8%.

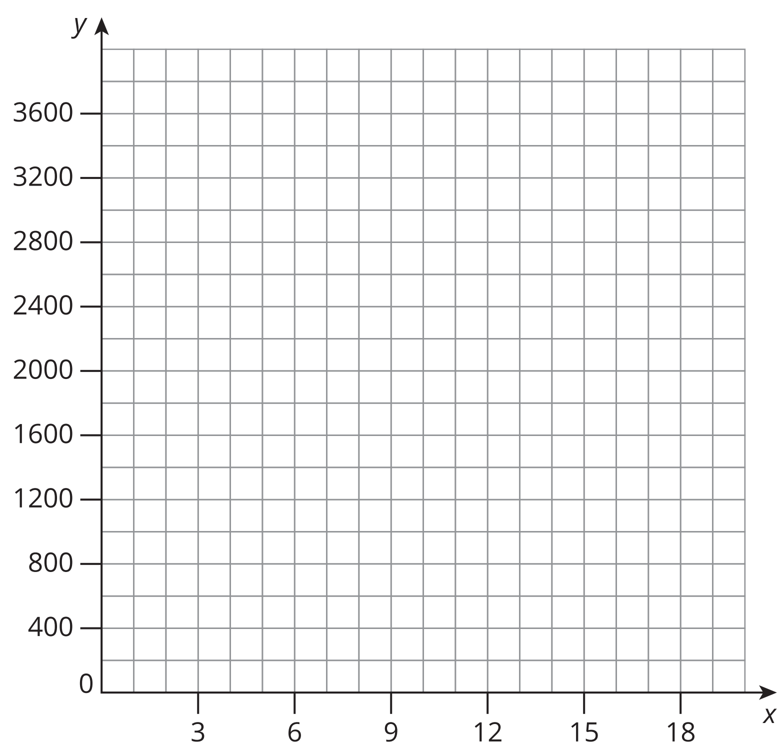
   f. A city has a population of 7000. Its population is decreasing at a rate of 1.25%.

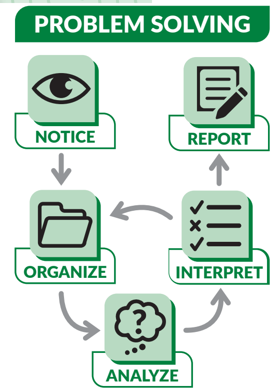
Talk the Talk

And More, Much More Than This . . .

A scientist is researching certain bacteria that have been found recently in the large animal cages at a local zoo. He starts with 200 bacteria that he intends to grow and study. He determines that every hour the number of bacteria increases by 25%.

1. Write a function and sketch a graph to represent this problem situation. Then, estimate the number of hours the scientist should let the bacteria grow to have no more than 2000 bacteria.





Review the steps and questions from the problem-solving model.

Ask Yourself . . .
Can others follow and understand your process and reasoning?

Lesson 1 Assignment

Write

Explain the difference between simple interest and compound interest.

Remember

An exponential growth function has a *b*-value greater than 1 and is of the form $y = a(1 + r)^x$, where *r* is the rate of growth. An exponential decay function has a *b*-value greater than 0 and less than 1 and is of the form $y = a(1 - r)^x$, where *r* is the rate of decay.

Practice

1. Valentina just received a $2500 bonus check from her employer. She is going to put it into an account that will earn interest. The Basic savings account at her bank earns 6% annual simple interest. The Gold savings account earns 4.5% interest compounded annually.

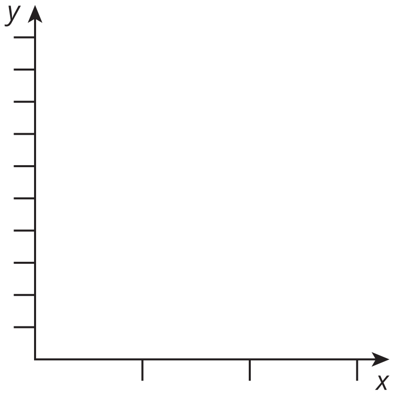
   a. Write a function for each account that can be used to determine the balance in the account based on the year, *t*. Describe each function.

Lesson 1 Assignment

b. Use your answers from part (a) to create a table of values for each function.

Time (years)	Basic Account Balance ($)	Gold Account Balance ($)

c. Use technology to graph the functions for the Basic and Gold savings accounts. Then, sketch the graphs.



d. Into which account would you recommend that Valentina deposit her money? Explain your reasoning.

e. After reading the pamphlet about the different accounts a little more closely, Valentina realizes that there is a one-time fee of $300 for depositing her money in the Gold account. Does this change the recommendation you made in part (d)? Why or why not?

Lesson 1 Assignment

f. Compare the rates of change for the Basic and Gold savings accounts. Explain what the rates of change tell you about the accounts.

g. What do the rates of change for linear and exponential functions tell you about the graphs of the functions?

2. Diego works for the owners of a bookstore. His starting salary was $24,500, and he gets a 3% raise each year.

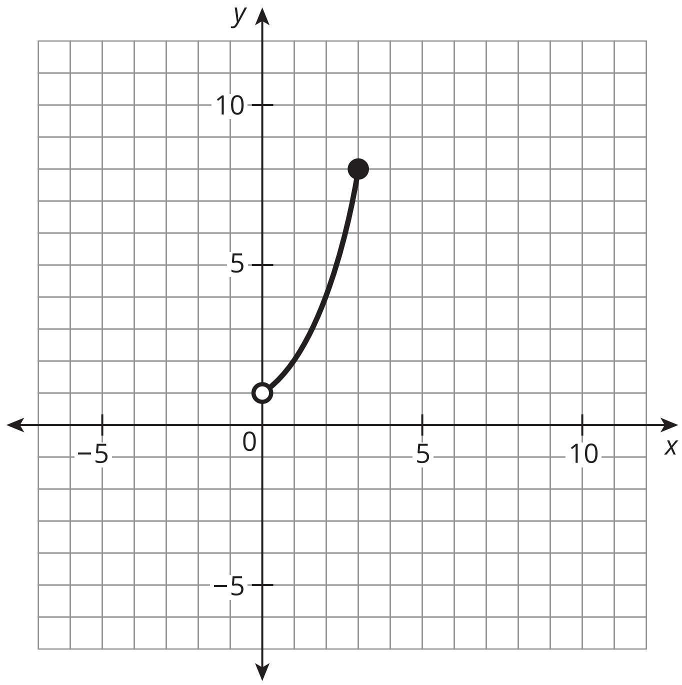
   a. Write an equation in function notation to represent Diego's salary as a function of the number of years he has been working at the bookstore.

   b. What will Diego's salary be when he begins his fourth year working at the bookstore? Show your work.

Lesson 1 Assignment

Prepare

A partial exponential function is graphed on the coordinate plane.



1. Identify the domain and range of the graph.

2 Interpreting Parameters in Context

OBJECTIVES

- Analyze equations and graphs of exponential functions.
- Match equations and graphs of exponential functions using the horizontal asymptote.
- Write and interpret exponential growth and decay functions.
- Use the properties of exponents to rewrite exponential functions.

You have written exponential functions for problem situations.

What strategies can you use to write and solve exponential equations?

Getting Started

Match Game

1. Match each graph with the correct function.

   Functions:

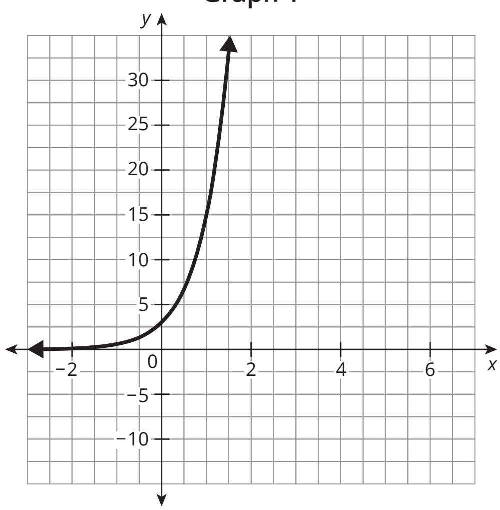
   $f(x) = 3(5)^x$

   $f(x) = 5(3)^x$

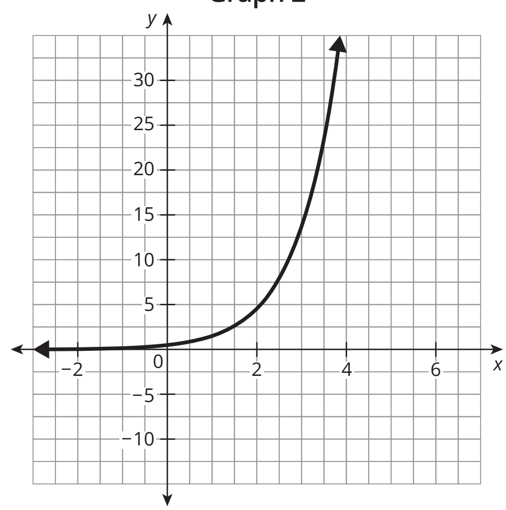
   $f(x) = 3\left(\frac{1}{2}\right)^x$

   $f(x) = \frac{1}{2} \cdot 3^x$

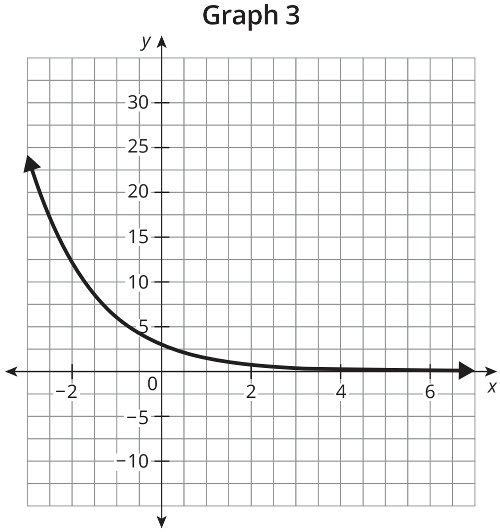
Graph 1



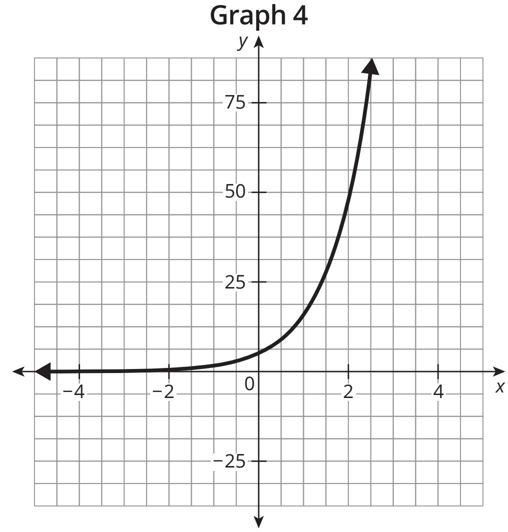
Graph 2



Graph 3



Graph 4



2. Sketch and label the y-intercept and asymptote on each graph.

3. Identify the domain and range of each graph.

Graph 1

Graph 2

Graph 3

Graph 4

4. Match each scenario with the correct function.

Scenario A: Nahimana spends one-half writing today. From day to day, she plans to triple the amount of time she spends writing. How much time, in hours, will Nahimana spend writing at the end of *x* days?

Scenario B: Luna has 3 hats in her collection. Each year, she plans to purchase 5 times the number of hats in her collection from the previous year. How many hats will Luna have at the end of *x* years?

Scenario C: Jasmine has 5 collectible figurines. Each year, she plans to triple the number of figurines she has in her collection from the previous year. How many figurines will Jasmine have at the end of *x* years?

Scenario D: Lucas spends 3 hours walking today. Each day, he wants to spend more time running so he plans to reduce the amount of time he spends walking each day by $\frac{1}{2}$ from the previous day. How much time will Lucas spend walking at the end of *x* days?

ACTIVITY

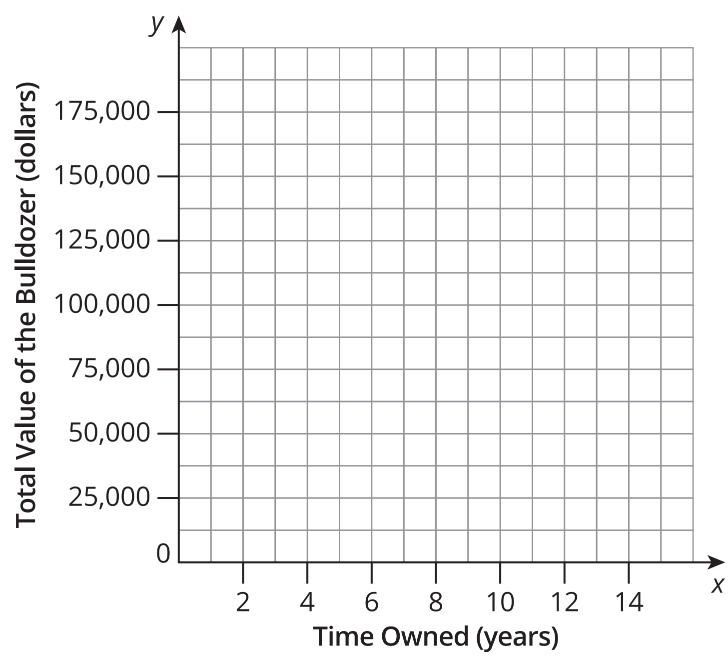
2.1 Solving Exponential Equations by Graphing

Depreciation is a decline in the value of something. Vehicles usually depreciate over time, meaning their value decreases over time. This decrease can often be represented by an exponential decay function.

A construction company bought a new bulldozer for $125,000. The bulldozer depreciates exponentially, and after 2 years, the value of the bulldozer is $80,000.

1. Write a function to represent the value of the bulldozer as a function of the number of years it is owned. Then, complete the table and graph.

Number of Years Owned	Value of Bulldozer
0	
2.5	
5	
7	
8.5	
10	
12.5	



Ask yourself . . .
What does each point on the graph represent?

2. The company wants to sell the bulldozer and get at least $25,000 from the sale. Use the graph to estimate the amount of time the company has to achieve this goal.

3. Estimate when the bulldozer will be worth:

   a. $50,000.

   b. $10,000.

4. When will the bulldozer be worth $0?

5. Identify and interpret each key feature in terms of the problem situation.

   a. y-intercept

   b. Asymptote

   c. x-intercept

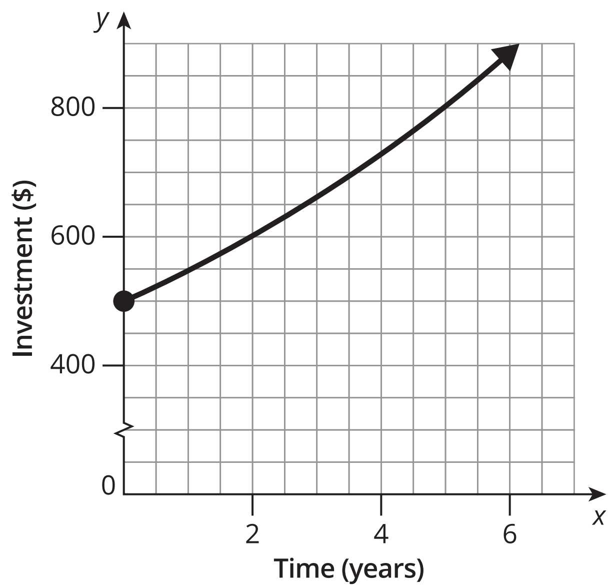
ACTIVITY 2.2

Interpreting the *b*-value

Adriana has invested $500 in a mutual fund which has shown an annual increase of about 10%.

1. Write a function, $f(t)$, that represents Adriana's investment in terms of t, time in years.

   The graph also represents Adriana's investment in terms of t, time in years.



2. Why is the graph a partial exponential function?

3. Use the graph to write the domain and range of the problem situation.

Suppose Adriana is interested in determining the monthly rate of increase. What is the approximate equivalent monthly rate of increase for her mutual fund?

4. Consider the responses from two of Adriana's friends. Describe the differences in their reasoning and why Rahsaan is correct.

Emily

Because we are dividing up the annual rate of increase over twelve months, divide the constant ratio by 12.

$$\frac{1.10}{12}$$

Logan

Because the annual rate of increase is represented as a multiplier, take the 12th root of the constant ratio.

$$1.10^{\frac{1}{12}}$$

Avery wants to write a function that is equivalent to the annual rate of increase but reveals the monthly rate of increase.

5. Explain why Avery's reasoning is not correct.

Avery

Since Adriana's monthly rate of increase is the twelfth root of the annual rate of increase, I can use the function

$$f(x) = 500\left(1.10^{\frac{1}{12}}\right)^{t}.$$

To rewrite the function representing Adriana's annual increase as an equivalent function that reflects the monthly rate of increase, you must change the *b*-value. The *b*-value of an exponential function can be written as the coefficient of *x*.

$$f(x) = a(b)^{Bx}$$

WORKED EXAMPLE

You can use what you know about common bases to rewrite the expression in an equivalent form.

$\left(1.10^{\frac{1}{12}}\right)^{Bx} = (1.10)^x$

$(1.10)^{\frac{Bx}{12}} = (1.10)^x$	Apply the power to a power rule.
$\frac{Bx}{12} = x$	The bases are the same, so the exponents must be equivalent expressions.
$Bx = 12x$	Multiply both sides by 12.
$B = 12$	Divide both sides by *x*.

So, the function $f(x) = 500\left(1.10^{\frac{1}{12}}\right)^{12x}$ is equivalent to the function $f(x) = 500(1.10)^x$.

6. Suppose Adriana wants to determine how much her mutual fund increases each quarter. Rewrite the original function in an equivalent form that reveals the approximate equivalent quarterly rate of increase.

7. What is Adriana's monthly increase, as a percent?

Talk the Talk

Is This Uptown or Downtown?

In 2005, the population of a city was 42,500. By 2010, the population had grown to approximately 51,708 people.

1. Identify any equations that are appropriate exponential models for the population of the city. Explain why. Then, explain why the equations you did not choose are not appropriate models for the situation.

$f(t) = 51{,}708(1.04)^t$

$f(t) = 51{,}708(0.96)^t$

$f(t) = 42{,}500(1.04)^{5t}$

$f(t) = 51{,}708(1.04)^{\frac{1}{5}t}$

$f(t) = 42{,}500(1.04)^t$

$f(t) = 42{,}500(0.96)^t$

Lesson 2 Assignment

Write

Explain how an asymptote can be identified from an exponential equation and its graph.

Remember

You can estimate the solution to an exponential equation graphically. First, graph both the exponential function and the constant function for the given *y*-value. Next, determine the point of intersection of the graph of the exponential function and the horizontal line. Lastly, identify the *x*-value of the coordinate pair as the solution.

Practice

1. Sarah bought a brand new car for $18,000. Its value depreciated at a rate of 1.2%.

   a. Write a function to represent the value of the car as a function of time. Use technology to estimate the number of years it will take for the value to reach each given amount.

   b. $17,000

   c. $15,000

   d. One-half of the starting value

   e. One-third the starting value

   f. $0

   g. $10,000

Lesson 2 Assignment

2. In 2012, the population of a city was 63,000. By 2017, the population was reduced to approximately 54,100. Identify any equations that are appropriate models for the population of the city, and explain why the others are not.

   a. $f(x) = 63{,}000(1.03)^{t}$

   b. $f(x) = 54{,}100(1.03)^{t}$

   c. $f(x) = 63{,}000(0.97)^{t}$

   d. $f(x) = 54{,}100(0.97)^{t}$

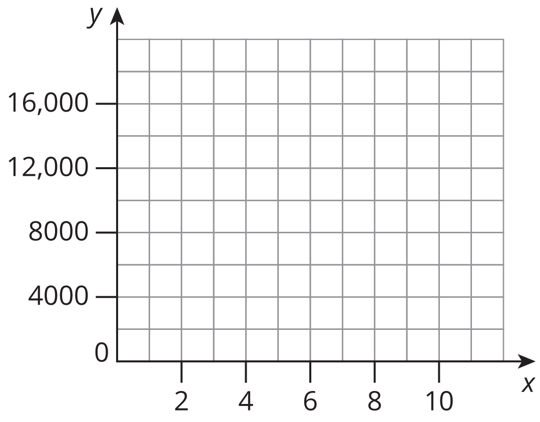
   e. $f(x) = 63{,}000(0.97)^{\frac{1}{5t}}$

   f. $f(x) = 54{,}100(0.97)^{5t}$

3. Camilla wants to own a bee colony so that she can extract honey from the hive. She starts a colony with 5000 bees. The number of bees grows exponentially with a growth factor of 12% each month.

   a. Write a function, $f(x)$, for the bee population that can be used to determine the number of bees in the colony, based on the month, x.

   b. Use technology to graph the function, $f(x)$.



Lesson 2 Assignment

c. What is the domain and range of the problem situation?

d. Identify and interpret the *y*-intercept.

e. Camilla feels that to get a decent amount of honey, there should be at least 15,000 bees in the colony. Estimate how many months it will take Camilla until she has 15,000 bees.

Prepare

While driving to their vacation spot, the Williams family kept a record of their gas purchases. Their information is recorded in the table.

Amount of Gas (gallons)	Total Cost (dollars)
9	21.33
16.6	40.00
11.8	26.08
10	24.70
13.2	28.91

1. Use technology to write the linear regression function.

2. What is the correlation coefficient, *r*? What does this value imply?

3 Modeling Using Exponential Functions

OBJECTIVES

- Write an exponential function to model a table of values and a graph.
- Add an exponential function and a constant function.
- Write an exponential function to model a data set.
- Use exponential models to solve problems.

You can interpret exponential scenarios, equations, tables, and graphs.

How can you use an exponential function to model real-world data?

Getting Started

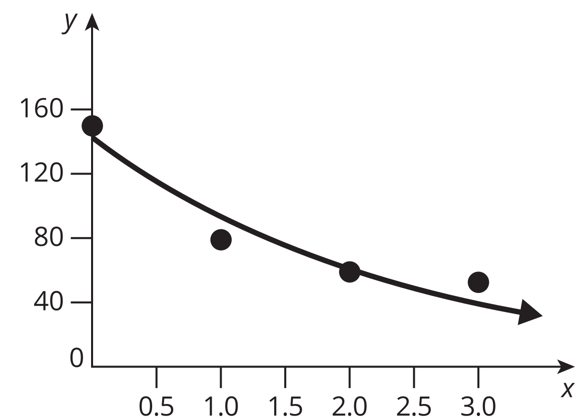
The Elephants in the Room

Two elephant populations over time are shown, each within a different small area in Africa.

Population A

Time (years)	Elephant Population
3	3218
5	3628
7	3721
9	3871

Population B



1. Estimate an exponential function for each population change over time. Explain how you determined your functions.

2. Use your functions and the features of each situation to describe the change in the populations over time.

ACTIVITY 3.1

Modeling a Decreasing Exponential Function

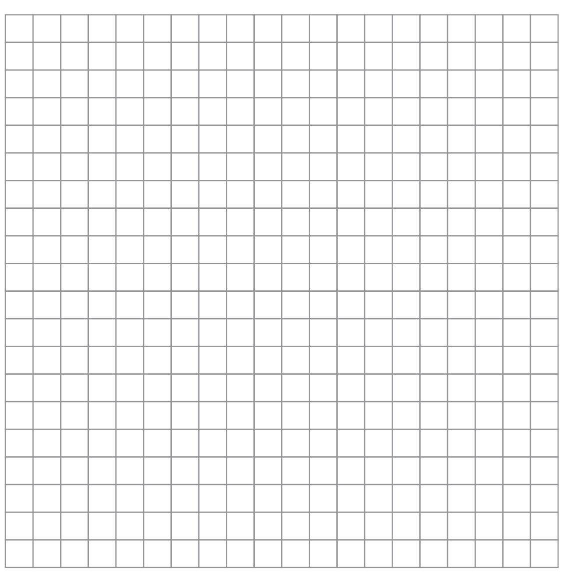
Avery loves drinking green tea. One morning, after making herself a cup of hot tea, she sat in her kitchen to enjoy it.

The table shows the temperature of a cup of Avery's tea over time.

Time (minutes)	Temperature (degrees Fahrenheit)
0	180
5	169
11	149
15	142
18	135
25	124
30	116
34	113
42	106
45	102
50	101

1. Model this situation.

   a. Create a scatterplot. Label your axes.



b. Use technology to write the exponential regression function. Define the variables. Identify the correlation coefficient.

c. Sketch the function on the same graph as your scatterplot.

2. State the domain and range of the function you sketched. How do they compare to the domain and range of this problem situation?

3. Use the function to predict the temperature of Avery's tea after an hour.

4. Use the function to predict the temperature of Avery's tea after 4 hours.

5. Does your prediction make sense in terms of this problem situation? Explain your reasoning.

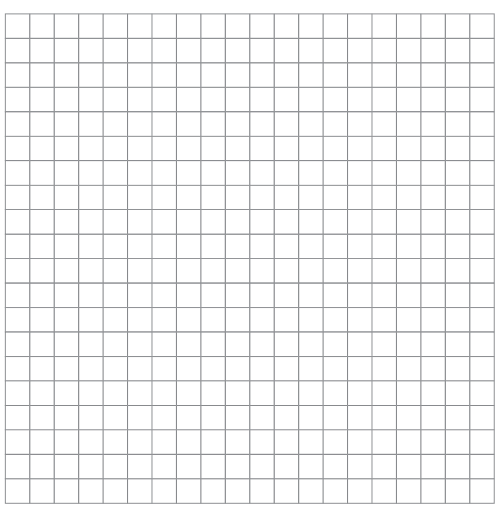
ACTIVITY 3.2 Determining the Best Fit to Model Data

The table shows the carbon dioxide concentration in Earth's atmosphere in parts per million from 1860 to 2017.

Year	Carbon Dioxide Concentration (parts per million)
1860	286
1880	291
1900	296
1920	303
1940	311
1960	317
1980	339
2000	370
2010	389
2017	406

1. Model this situation.

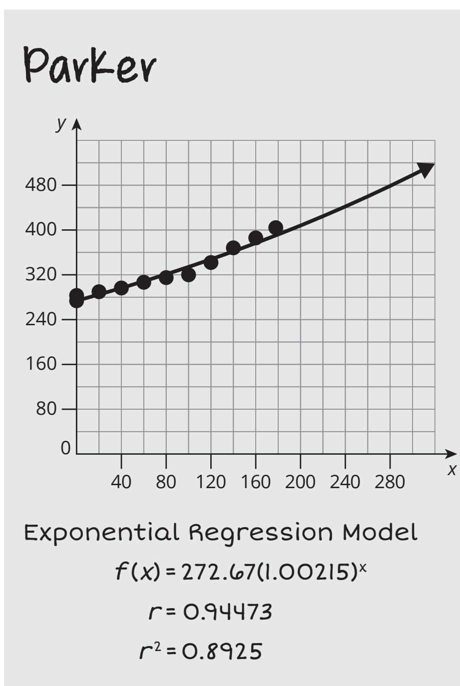
   a. Create a scatterplot using 1860 as Year 0. Label your axes.

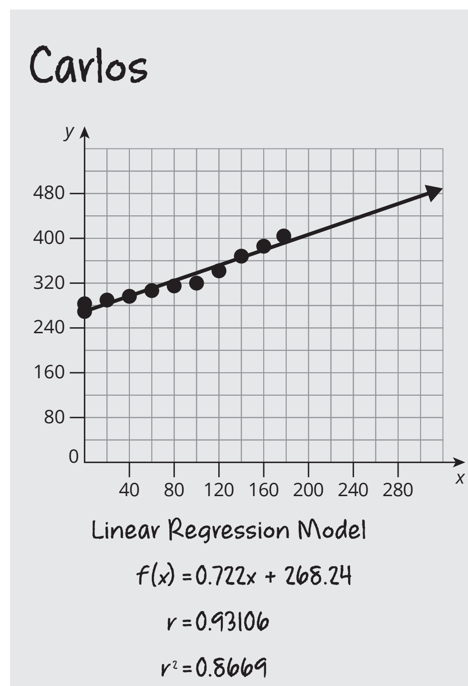


b. Use technology to choose the correct function to model this data. Write your regression function. Define the variables. Identify the correlation coefficient.

c. Sketch the function on the same graph as your scatterplot.

2. Parker used an exponential function to model the situation. Carlos used a linear function to model the situation. Who is correct? Explain your reasoning.





3. What other information would help you to make the decision as to whether a linear or exponential function is best to model this context and data?

Ask Yourself . . .
How can you use regression models in everyday life?

4. Why does the exponential function look very similar to the linear function?

5. Use each function to predict the concentration of carbon dioxide for each given year.

   a. During 2160

   b. During 2500

Talk the Talk

Making a List, Checking It Twice

Reflect on the exponential situations and graphs you have encountered in past lessons.

1. Consider the contexts.

   a. Describe the contexts that can be modeled by an exponential function.

   b. What do the contexts have in common that identify them as being exponential functions?

2. Consider the graphs.

   a. How can you tell from a scatterplot that it can be modeled by an exponential function?

   b. Sketch four different possible graphs of an exponential function of the form $y = ab^x$. Describe the a-value and common multiplier in each.

Lesson 3 Assignment

Write

Describe the information that can be used to determine whether a linear or exponential function is best to model a context and data.

Remember

You can use exponential functions to model scenarios that involve a percent increase or decrease, such as compound interest and population growth or decay.

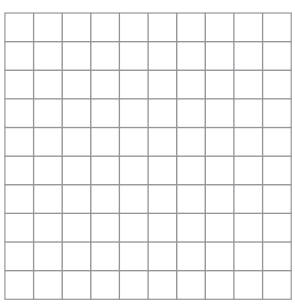
Sometimes, it may be difficult to determine whether a scatterplot is best modeled by a linear or exponential function. In these cases, sometimes knowing the scenario can help, while in other cases, more data points or information may be needed.

Practice

1. The table shows the number of U.S. Post Offices at the beginning of each decade from 1900 to 2000.

Year	Number of U.S. Post Offices
1900	76,688
1910	59,580
1920	52,641
1930	49,063
1940	44,024
1950	41,464
1960	35,238
1970	32,002
1980	30,326
1990	28,959
2000	27,876

   a. Create a scatterplot of the data.



   b. Determine the exponential regression function and the value of the correlation coefficient, *r*. Then, graph the equation on the grid with the scatterplot.

Lesson 3 Assignment

c. Predict the number of U.S. Post Offices in the year 2050.

d. Predict when the number of U.S. Post Offices will reach 20,000.

e. What do you think is causing the decline in the number of U.S. Post Offices?

Prepare

Consider $f(x) = x^2 + 3x + 4$. Evaluate the function for each given value.

1. $f(1)$
2. $f(-1)$
3. $f(2)$
4. $f(-2)$

TOPIC 2 SELF-REFLECTION

Name: ____________________

Using Exponential Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Using Exponential Equations* topic by:

TOPIC 2: *Using Exponential Equations*	Beginning of Topic	Middle of Topic	End of Topic
using graphs, tables, and equations to compare the output values of linear and exponential functions.	◯	◯	◯
interpreting exponential expressions by viewing one or more of their parts as a single entity.	◯	◯	◯
writing an exponential equation to model a real-world problem.	◯	◯	◯
graphing exponential equations that model growth and decay on a coordinate plane and using the graph to answer questions about the real-world situation that it models.	◯	◯	◯
explaining that every point on the graph of an exponential equation is a solution to that equation.	◯	◯	◯
using the intersection of a graph of an exponential function and a horizontal line to determine a solution to an exponential equation.	◯	◯	◯
explaining the meaning of the values of a and b in the exponential function $f(x) = ab^x$.	◯	◯	◯

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2: *Systems of Linear Equations and Inequalities*	Beginning of Topic	Middle of Topic	End of Topic
using technology to determine a curve of best fit for a set of data that appears to be exponential.	○	○	○
using an exponential regression model to make predictions for the real-world problem it represents.	○	○	○

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Using Exponential Equations* topic.

____

____

____

____

____

____

2. What mathematical understandings from the topic do you feel you are making the most progress with?

____

____

____

____

____

____

TOPIC 2 SELF-REFLECTION *continued*

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Using Exponential Equations Summary

LESSON 1 Exponential Equations for Growth and Decay

NEW KEY TERMS

- simple interest [interés simple]
- compound interest [interés compuesto]
- exponential growth function
- exponential decay function [función de decaimiento exponencial]

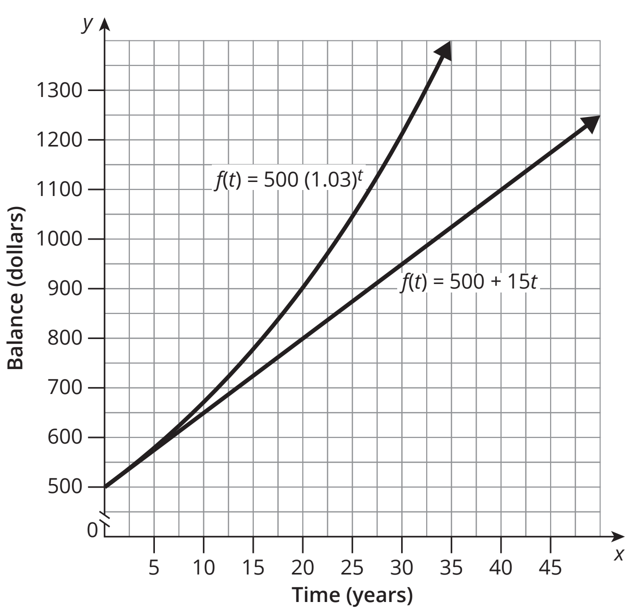
In a **simple interest** account, a percent of the starting balance is added to the account at each interval. The formula for simple interest is $I = Prt$, where P represents the starting amount, or principal, r represents the interest rate, t represents time, and I represents the interest earned.

In a **compound interest** account, the balance is multiplied by the same amount at each interval. Because the entire balance is multiplied by the same percent for each interval, the formula is represented by an exponential equation: $I = P(1 + r)^t$.

The rate of change for a simple interest account is constant. The rate of change between the values for the compound interest account is increasing as t becomes larger. A constant rate of change means that the graph of the linear equation is a straight line. An increasing rate of change means that the graph of the exponential function is a smooth curve.

For example, consider two accounts that each have an initial balance of $500. One account earns 3% simple interest each year, while the other earns 3% compound interest each year.

Time (years)	Simple Interest Balance (dollars)	Compound Interest Balance (dollars)
0	500	500
1	515	515
2	530	530.45
10	650	671.96
100	2000	9609.32

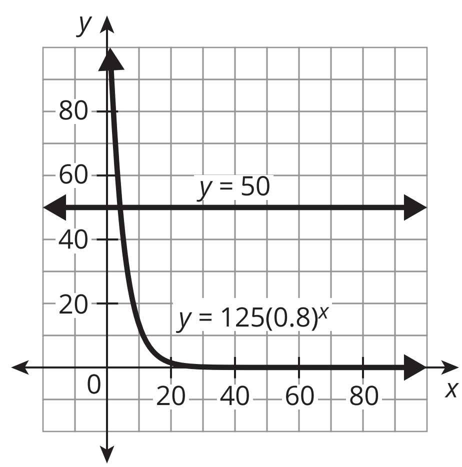


An **exponential growth function** has a b-value greater than 1 and is in the form $y = a(1 + r)^x$, where r is the rate of growth. An **exponential decay function** has a b-value greater than 0 and less than 1 and is in the form $y = a(1 - r)^x$, where r is the rate of decay.

For example, consider a population that starts at 50,000 and grows at a rate of 2% every year. The scenario can be modeled by the function $f(x) = 50{,}000(1.02)^x$. The a-value of the exponential function is the initial population and since the population grows, the b-value is $(1 + 0.02)$ or 1.02.

LESSON 2 Interpreting Parameters in Context

Graphs can be used to solve exponential equations by estimating the intersection point of the graph of an exponential function with a constant function.



For example, to determine the solution to $125(0.8)^x = 50$, graph each side of the equation as separate functions on the same coordinate plane.

The solution to the equation is $x \approx 4$.

LESSON 3

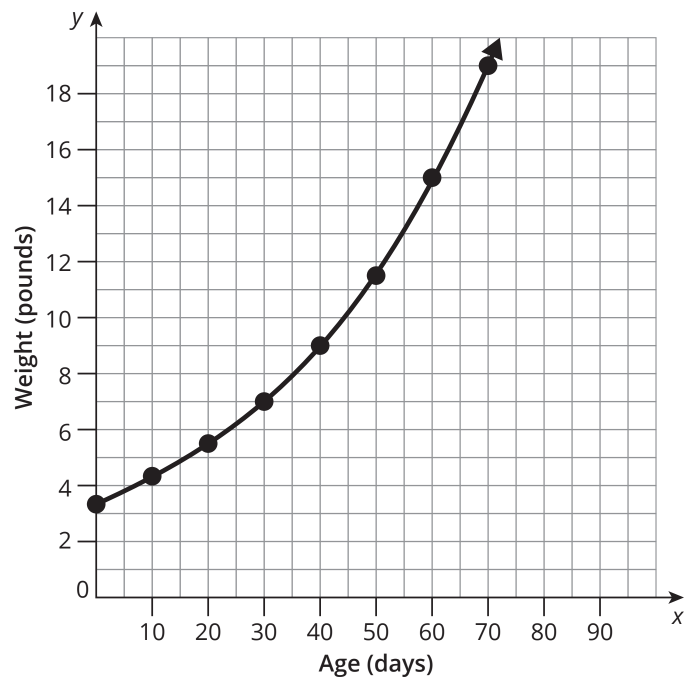
Modeling Using Exponential Functions

Technology can be used to determine which exponential regression function best models real-world data. This regression function can be used to predict future values.

For example, the table shows the weight of a golden retriever puppy as recorded during its growth. A scatterplot and regression function of the data are shown.

The exponential regression function for this scenario is $f(x) = 3.31(1.025)^x$. The correlation coefficient, r, is 0.9999.

Age (days)	Weight (pounds)
0	3.25
10	4.25
20	5.5
30	7
40	9
50	11.5
60	15
70	19



The equation can be used to predict the puppy's weight at 80 days.

$f(80) = 3.31(1.025)^{80}$
$f(80) = 23.9$

The puppy's weight will be approximately 23.9 pounds on day 80.

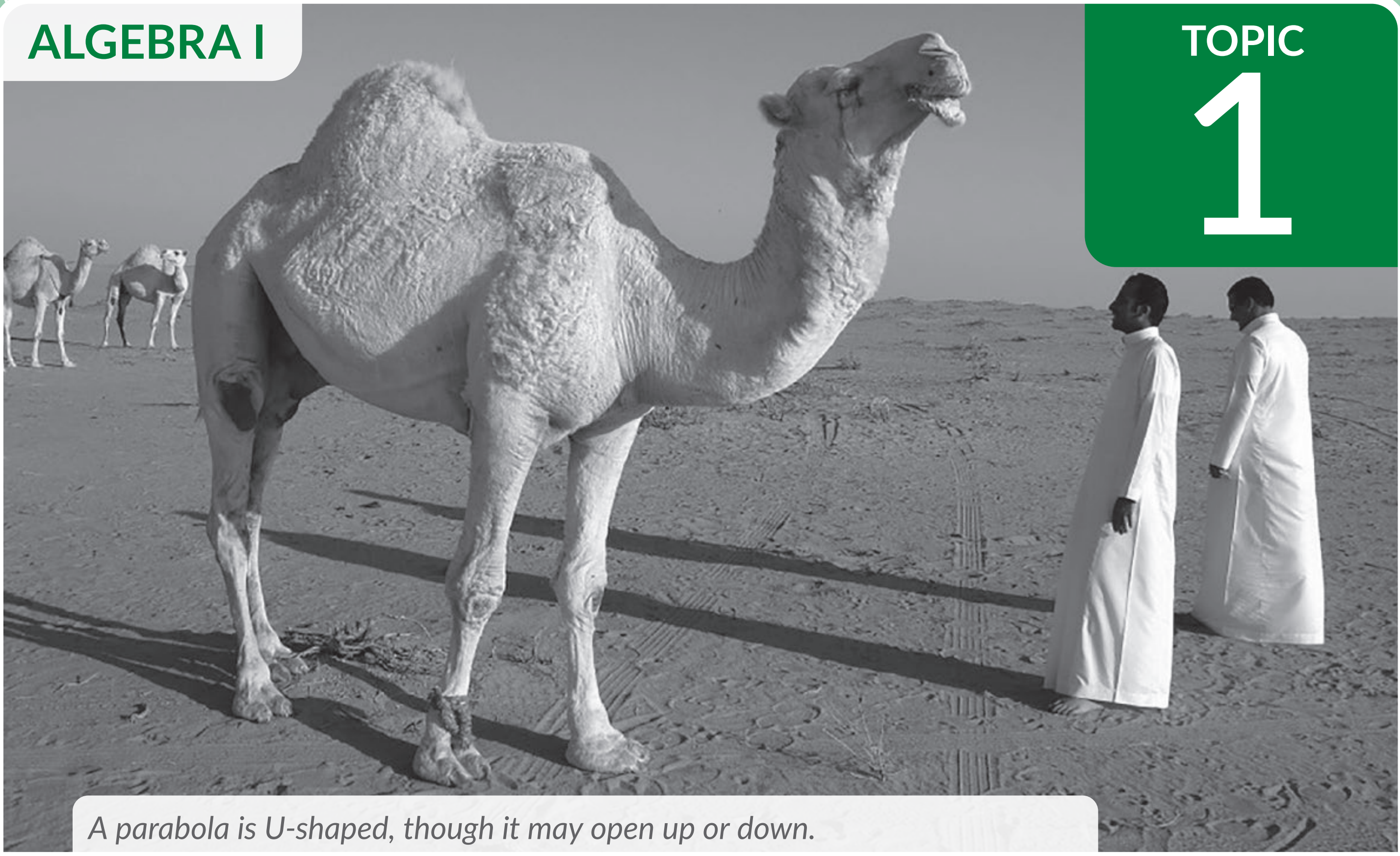
MODULE 5

Maximizing and Minimizing

TOPIC 1 Introduction to Quadratic Functions **657**

TOPIC 2 Polynomial Operations **757**

TOPIC 3 Solving Quadratic Equations **825**


A parabola is U-shaped, though it may open up or down.

Introduction to Quadratic Functions

LESSON 1 Exploring Quadratic Functions 659

LESSON 2 Key Characteristics of Quadratic Functions 675

LESSON 3 Quadratic Function Transformations 703

LESSON 4 Horizontal Transformations and Vertex Form 723

1 Exploring Quadratic Functions

OBJECTIVES

- Write quadratic functions to model contexts.
- Graph quadratic functions using technology.
- Interpret the key features of quadratic functions in terms of a context.
- Identify the domain and range of quadratic functions and their contexts.

NEW KEY TERMS

- parabola
- vertical motion model
- roots

You have used linear functions to model situations with constant change, and you have used exponential functions to model growth and decay situations.

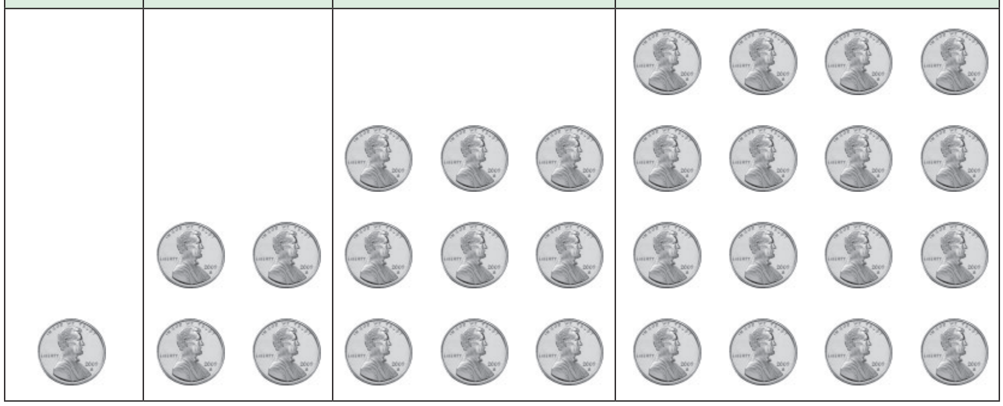
What type of real-world situations can you model using quadratic functions?

Getting Started

Squaring It Up

Destiny is using pennies to create a pattern.

Figure 1	Figure 2	Figure 3	Figure 4



1. Analyze the pattern and explain how to create Figure 5.

2. How many pennies would Destiny need to create Figure 5? Figure 6? Figure 7?

3. Which figure would Destiny create with exactly $4.00 in pennies?

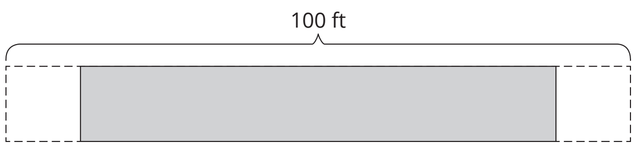
4. Write an equation to determine the number of pennies for any figure number. Define your variables.

5. Describe the function family to which this equation belongs.

ACTIVITY 1.1

Using Area to Introduce Quadratic Functions

A dog trainer is fencing in an enclosure, represented by the shaded region in the diagram. The trainer will also have two square-shaped storage units on either side of the enclosure to store equipment and other materials. She can make the enclosure and storage units as wide as she wants, but she can't exceed 100 feet in total length.



1. Let s represent a side length, in feet, of one of the storage units.

   a. Label the length and width of the enclosure in terms of *s*.

   b. Write the function $L(s)$ to represent the length of the enclosure as a function of side length, *s*.

   c. Sketch and label a graph of the function on the given coordinate plane. Identify any key points.

2. Describe the domain and range of the context and of the function.

Ask Yourself . . .
To identify key points on the graph, think about the function you are representing. Are there any intercepts? Are there any other points of interest?

The progression of diagrams below shows how the area of the enclosure, $A(s)$, changes as the side length, s, of each square storage unit increases.

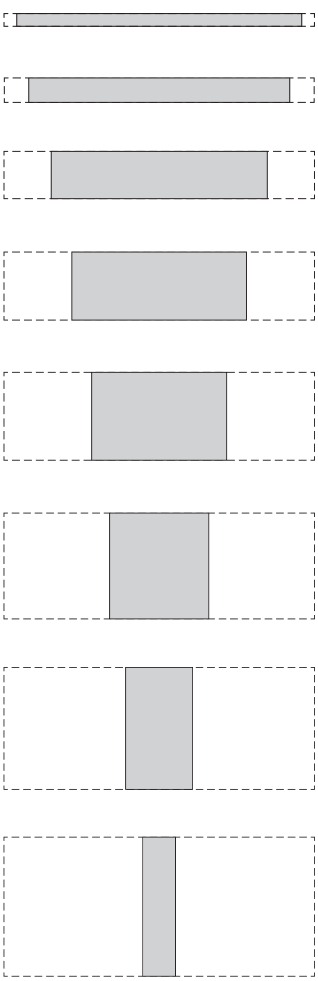
3. Identify each key characteristic of the graph. Then, interpret the meaning of each in terms of the context.

   a. Slope

   b. y-intercept

   c. Increasing or decreasing

   d. x-intercept



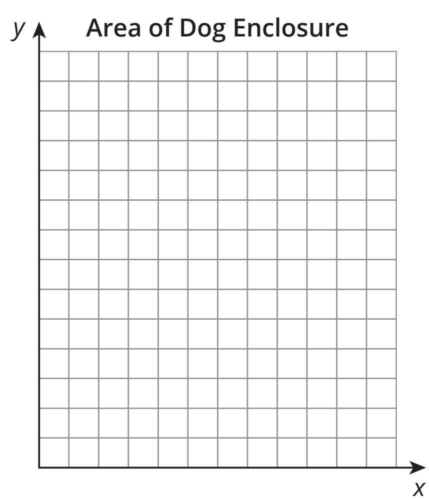
4. Write the function $A(s)$ to represent the area of the enclosure as a function of side length, s.

5. Describe how the area of the enclosure changes as the side length increases.

6. Consider the graph of the function, $A(s)$.

   a. Predict what the graph of the function will look like.

   b. Use technology to graph the function $A(s)$. Then, sketch the graph and label the axes.



7. Describe what all the points on the graph represent.

The function $A(s)$ that you wrote to model area is a quadratic function. The shape that a quadratic function forms when graphed is called a **parabola**.

Think about . . .
Quadratic functions model area because area is measured in square units.

8. Think about the possible areas of the enclosure.
   a. Is there a maximum area that the enclosure can contain? Explain your reasoning in terms of the graph and in terms of the context.
   b. Use technology to determine the maximum of $A(s)$. Describe what the x- and y-coordinates of the maximum represent in this context.
   c. Determine the dimensions of the enclosure that will provide the maximum area. Show your work and explain your reasoning.

9. Identify the domain and range of the context and of the function.

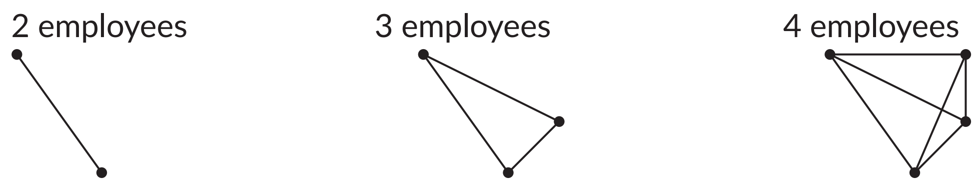
10. Identify each key characteristic of the graph. Then, interpret the meaning of each in terms of the context.
    a. y-intercept
    b. Increasing and decreasing intervals
    c. Symmetry
    d. x-intercepts

ACTIVITY
1.2

Writing and Interpreting a Quadratic Function

Suppose that there is a monthly meeting at CIA headquarters for all employees. How many handshakes will it take for every employee at the meeting to shake the hand of every other employee at the meeting once?

1. Use the figures shown to determine the number of handshakes that will occur between 2 employees, 3 employees, and 4 employees.



Ask Yourself . . .
Can you tell what shape the graph will be?

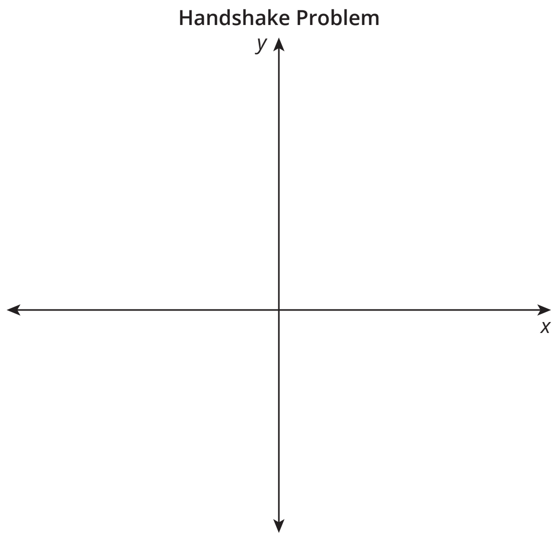
2. Draw figures to represent the number of handshakes that occur between 5 employees, 6 employees, and 7 employees, and determine the number of handshakes that will occur in each situation.

3. Enter your results in the table.

Number of Employees	2	3	4	5	6	7	n
Number of Handshakes							

4. Write a function to represent the number of handshakes given any number of employees. Enter your function in the table.

5. Use technology to graph the function you wrote in Question 4. Sketch the graph and label the axes.



6. How is the orientation of this parabola different from the parabola for the area of the dog enclosure? How is this difference reflected in their corresponding equations?

7. Determine the minimum of your function. Then, describe what the *x*- and *y*-coordinates of this minimum represent in this problem situation.

8. Identify the domain and range of the problem situation and of the function.

ACTIVITY 1.3 Using a Quadratic Function to Model Vertical Motion

You can model the motion of a pumpkin released from a catapult using a vertical motion model. A **vertical motion model** is a quadratic equation that models the height of an object at a given time. The equation is of the form shown.

$$y = -16t^2 + v_0t + h_0$$

In this equation, y represents the height of the object in feet, t represents the time in seconds that the object has been moving, v_0 represents the initial vertical velocity (speed) of the object in feet per second, and h_0 represents the initial height of the object in feet.

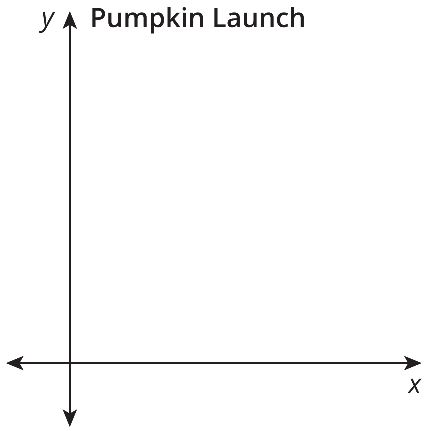
1. What characteristics of this situation indicate that you can model it using a quadratic function?

Suppose that a catapult hurls a pumpkin from a height of 68 feet at an initial vertical velocity of 128 feet per second.

2. Write a function for the height of the pumpkin, $h(t)$, in terms of time, t.

3. Does the function you wrote have a minimum or maximum? How can you tell from the form of the function?

4. Use technology to graph the function. Sketch your graph and label the axes.



Ask Yourself . . .
What do all the points on this graph represent?

5. Use technology to determine the maximum or minimum and label it on the graph. Explain what it means in terms of the problem situation.

Ask Yourself . . .
How can you use a maximum value of a function in everyday life?

6. Determine the *y*-intercept and label it on the graph. Explain what it means in terms of the problem situation.

7. Use a horizontal line to determine when the pumpkin reaches each height after being catapulted. Label the points on the graph.

   a. 128 feet

   b. 260 feet

   c. 55 feet

Remember . . .
The zeros of a function are the x-values when the function equals 0.

8. Explain why the x- and y-coordinates of the points where the graph and each horizontal line intersects are solutions.

9. When does the catapulted pumpkin hit the ground? Label this point on the graph. Explain how you determined your answer.

The time when the pumpkin hits the ground is one of the x-intercepts, $(x, 0)$. When an equation is used to model a situation, the x-coordinate of the x-intercept is referred to as a *root*. The **root** of an equation indicates where the graph of the equation crosses the x-axis.

ACTIVITY 1.4

Expressing a Quadratic Function as the Product of Two Linear Functions

The Alamo is a historic landmark in San Antonio, Texas that is now a museum. The museum charges $50 per tour, using the proceeds for maintenance and repairs of the Alamo property. Each summer, he books 100 tours at that price. Michael is considering a decrease in the price per tour because he thinks it will help him book more tours. He estimates that he will gain 10 tours for every $1 decrease in the price per tour.

1. According to the scenario, how much money does the museum currently generate each summer with their tours?

Revenue is the amount of money regularly coming into a business. At the Alamo, the revenue is the number of tours multiplied by the price per tour. Your response to Question 1 can be referred to as revenue. Because Michael is considering different numbers of tours and prices per tour, the revenue can be modeled by a function.

2. Write a function, $r(x)$, to represent the revenue for the Alamo.

   a. Let x represent the decrease in the price per tour. Write an expression to represent the number of tours booked when the decrease in price is x dollars per tour.

   b. Write an expression to represent the price per tour when Michael decreases the price x dollars per tour.

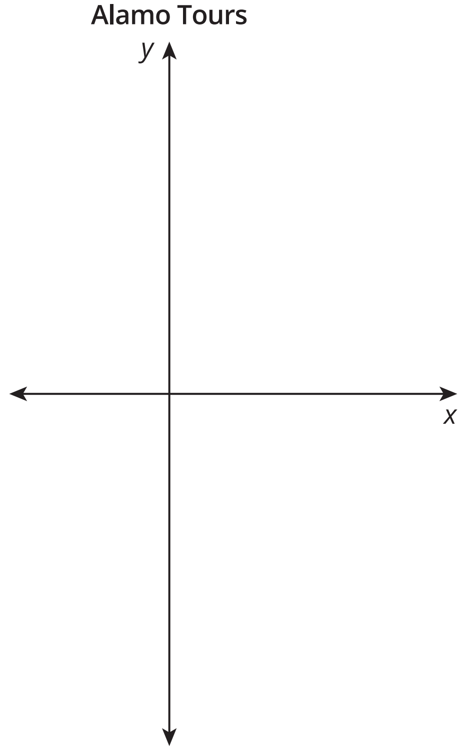
Think About . . .
You can always check your function by testing it with values of x. What is the value of $r(x)$ when $x = 0$? Does it make sense?

c. Use your expressions from parts (a) and (b) to represent the revenue, $r(x)$, as the number of tours times the price per tour.

Revenue	=	Number of Tours	·	Price per Tour
______	=	______________	·	______________

3. Use technology to graph the function $r(x)$. Sketch your graph and label the axes.

Remember . . .
Don't forget to label key points!



Ask Yourself . . .
What tools or strategies can you use to solve this problem?

4. Assume that Michael estimates that for every $1 decrease in the price per tour, he will gain 10 tours is accurate.

   a. What is the maximum revenue that the Alamo could earn for the summer?

b. Ana and Javier are calculating the number of tours that would yield the maximum revenue.

Ana said that according to the graph, a tour should cost \$20. Since $\$9000 \div \$20 = 450$, the number of tours would be 450.

Javier said that the cost of a tour should be \$30, and \$9000 divided by \$30 per tour is 300 tours.

Who is correct? Explain your reasoning.

c. Would you advise Michael to adjust the cost per tour to make the maximum revenue? Why or why not?

5. Identify each key characteristic of the graph. Then, interpret its meaning in terms of the context.

a. *x*-intercepts

b. *y*-intercept

c. Increasing and decreasing intervals

Talk the Talk

Making Connections

Analyze the graphs of the four quadratic functions in this lesson.

1. Summarize what you know about the graphs of quadratic functions. Include a sketch or sketches and list any characteristics.

2. Compare your sketch or sketches and list with your classmates. Did you all sketch the same parabola? Why or why not?

Lesson 1 Assignment

Write

Fill in the blank.

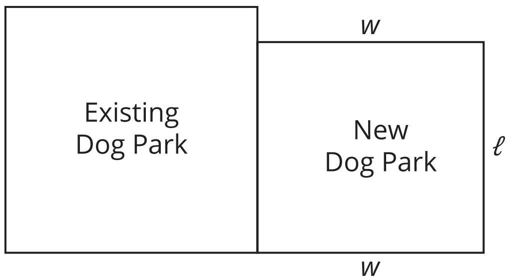
1. The x-intercepts of a graph of a quadratic function are the ______________ of the quadratic function.
2. A quadratic equation that models the height of an object at a given time is a ________________.
3. The shape that a quadratic function forms when graphed is called a ______________.
4. The ______________ of an equation indicate where the graph of the equation crosses the x-axis.

Remember

The graph of a quadratic function is called a *parabola*. Parabolas are smooth curves that have an absolute maximum or minimum, both increasing and decreasing intervals, up to two x-intercepts, and symmetry.

Practice

1. The citizens of a local community have an existing dog park for dogs to play, but they have decided to build another one so that one park will be for small dogs and the other will be for large dogs. The plan is to build a rectangular fenced-in area adjacent to the existing dog park, as shown in the sketch. The county has enough money in the budget to buy 1000 feet of fencing.



   a. Determine the length of the new dog park, ℓ, in terms of the width, w.

   b. Write the function $A(w)$ to represent the area of the new dog park as a function of the width, w. Does this function have a minimum or a maximum? Explain your answer.

Lesson 1 Assignment

c. Sketch the graph of the function. Label the axes, the maximum or minimum, the x-intercepts, and the y-intercept.

d. Determine the x-intercepts of the function. Explain what each x-intercept means in terms of the problem situation.

e. What should the dimensions of the dog park be to maximize the area? What is the maximum area of the park?

f. Use the graph to determine the dimensions of the park when you restrict the area to 105,000 square feet.

Prepare

Determine the slope and y-intercept of each linear function.

1. $h(x) = 3x$

2. $g(x) = \frac{1}{2}(x - 5)$

3. $k(x) = x - 2$

4. $m(x) = \frac{8x}{4} + 1$

2 Key Characteristics of Quadratic Functions

OBJECTIVES

- Identify the factored form and standard form of an equation for a quadratic function.
- Determine the equation for the axis of symmetry of a quadratic function given the equation in standard form or factored form.
- Determine the absolute minimum or absolute maximum point on the graph of a quadratic function and identify this point as the vertex.
- Describe intervals of increase and decrease in relation to the axis of symmetry on the graph of a quadratic function.
- Use key characteristics of the graph of a quadratic function to write an equation in factored form.

NEW KEY TERMS

- second differences
- concave up
- concave down
- standard form of a quadratic function
- factored form
- vertex of a parabola
- axis of symmetry

You have identified key characteristics of linear and exponential functions.

What are the key characteristics of quadratic functions?

Getting Started

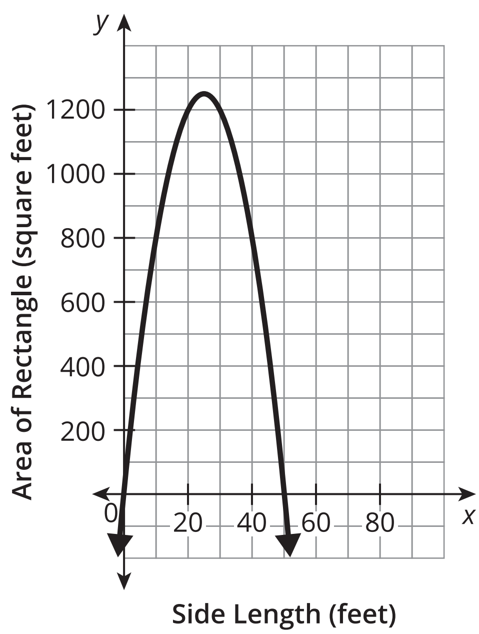
Dogs, Handshakes, Pumpkins, and the Alamo

Consider the four quadratic models you investigated in the previous lesson. There are multiple equivalent ways to write the equation to represent each situation and a unique parabola to represent the equivalent equations. You can also represent the function using a table of values.

Area of Dog Enclosure

$A(s) = -2s^2 + 100s$

$\quad = -2(s)(s - 50)$

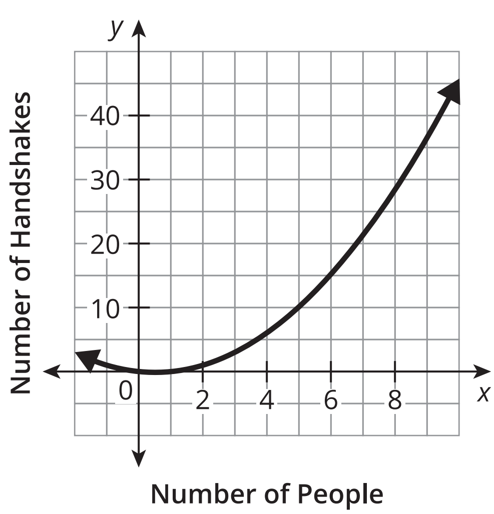


s	$A(s)$
0	0
1	98
2	192
3	282
4	368

Handshake Problem

$f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$

$\quad = \frac{1}{2}(n)(n - 1)$

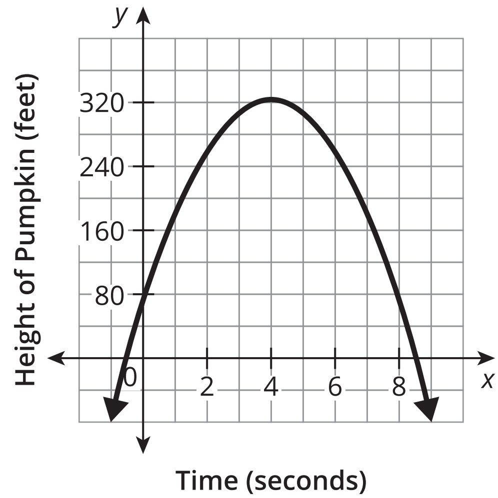


n	$f(n)$
0	0
1	0
2	1
3	3
4	6

Pumpkin Launch

$h(t) = -16t^2 + 128t + 68$

$= -16\left(t - \frac{17}{2}\right)\left(t + \frac{1}{2}\right)$

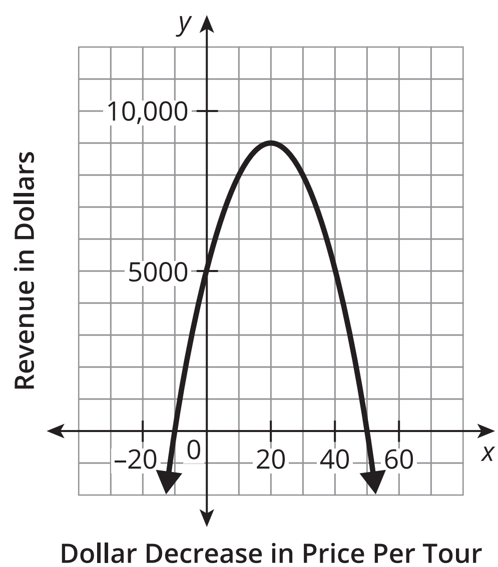


t	$h(t)$
0	68
1	180
2	260
3	308
4	324

Alamo Tours

$r(x) = -10(x + 10)(x - 50)$

$= -10x^2 + 400x + 5000$



x	$r(x)$
0	5000
1	5390
2	5760
3	6110
4	6440

1. Consider each representation.

   a. How can you tell from the structure of the equation that it is quadratic?

   b. What does the structure of the equation tell you about the shape and characteristics of the graph?

c. How can you tell from the shape of the graph that it is quadratic?

d. How can you tell from the table that the relationship is quadratic?

ACTIVITY **2.1**

Second Differences

Let's explore how a table of values can show that a function is quadratic. Consider the table of values represented by the basic quadratic function. This table represents the first differences between seven consecutive points.

x	$f(x)$	First Differences
−3	9	
		$4 - 9 = -5$
−2	4	
		$1 - 4 = -3$
−1	1	
		$0 - 1 = -1$
0	0	
		$1 - 0 = 1$
1	1	
		$4 - 1 = 3$
2	4	
		$9 - 4 = 5$
3	9	

Remember . . .

You can tell whether a table represents a linear function by analyzing first differences. First differences imply the calculation of $y_2 - y_1$.

1. What do the first differences tell you about the relationship of the table of values?

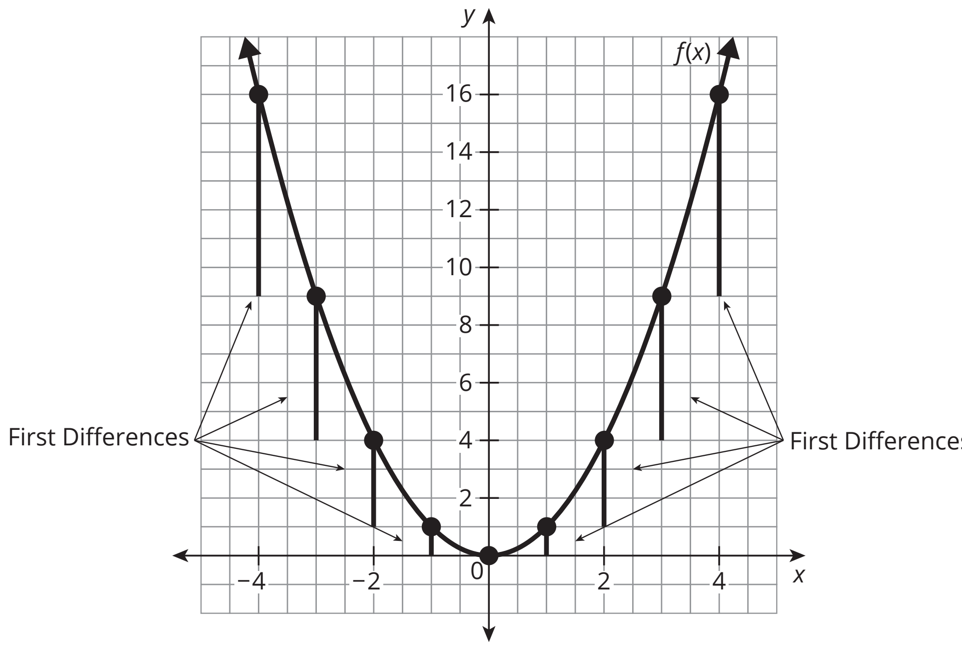
Let's consider the *second differences*. The **second differences** are the differences between consecutive values of the first differences.

2. Calculate the second differences for $f(x)$. What do you notice?

You know that with linear functions, the first differences are constant. For quadratic functions, the second differences are constant.

Let's consider the graph of the basic quadratic function, $f(x) = x^2$, and the distances represented by the first and second differences. Graph 1 shows the distances between consecutive values of $f(x)$. The colored line segments are different lengths because the first differences are not the same.

Graph 1

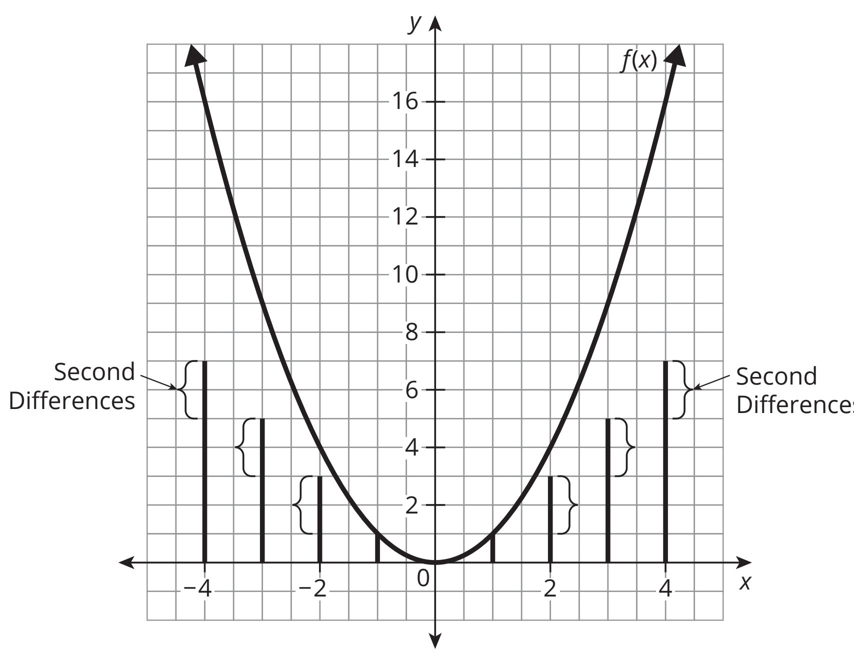


Think About . . .

Quadratic equations are polynomials with a degree of 2. Their second differences are constant. Linear functions are polynomials with a degree of 1, and their first differences are constant.

Graph 2 shows the lengths of the first differences positioned along the x-axis. By comparing these lengths, you can see the second differences.

Graph 2



3. How does the representation in Graph 1 support the first differences calculated from the table of values?

4. How does the representation in Graph 2 support the second differences you calculated in the table?

5. Identify each equation as linear or quadratic. Complete the table to calculate the first and second differences. Then, sketch the graph.

a. $y = 2x$ ____________

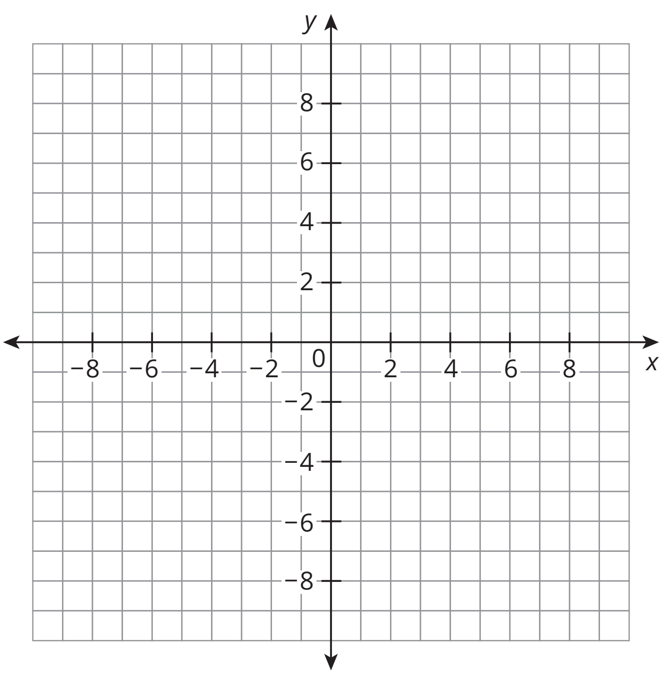
x	y
−3	−6
−2	−4
−1	−2
0	0
1	2
2	4
3	6

First Differences

_____
_____
_____
_____
_____
_____

Second Differences

_____
_____
_____
_____
_____



b. $y = 2x^2$ ____________

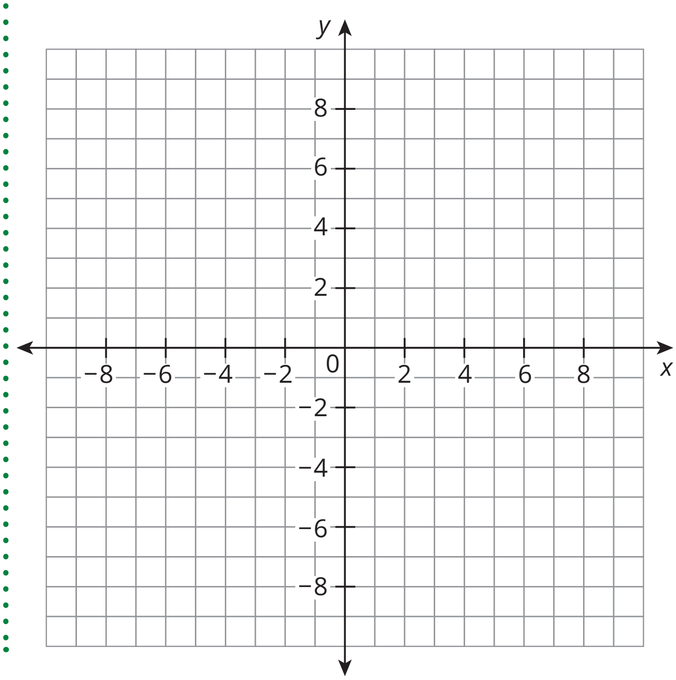
x	y
−3	18
−2	8
−1	2
0	0
1	2
2	8
3	18

First Differences

_____
_____
_____
_____
_____
_____

Second Differences

_____
_____
_____
_____
_____

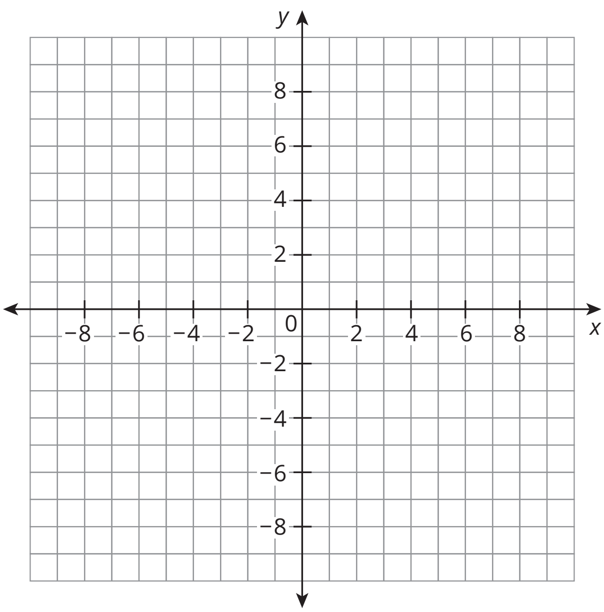


c. $y = -x + 4$ ____________

x	y
−3	7
−2	6
−1	5
0	4
1	3
2	2
3	1

First Differences: _____ _____ _____ _____ _____ _____

Second Differences: _____ _____ _____ _____ _____

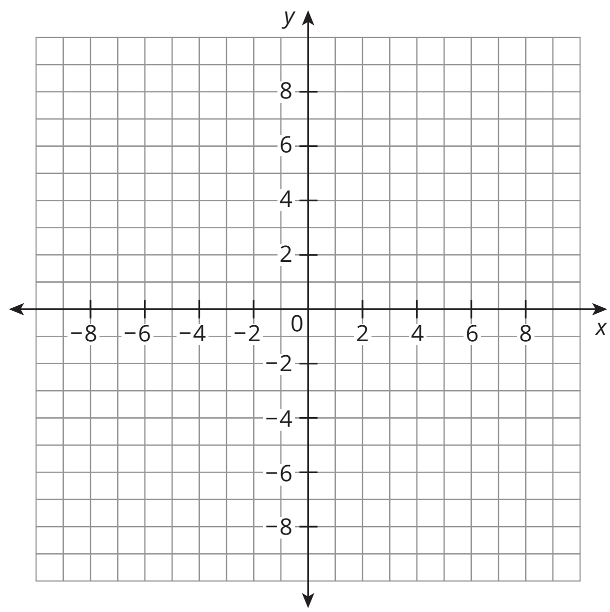


d. $y = -x^2 + 4$ ____________

x	y
−3	−5
−2	0
−1	3
0	4
1	3
2	0
3	−5

First Differences: _____ _____ _____ _____ _____ _____

Second Differences: _____ _____ _____ _____ _____



A graph that opens upward is identified as being **concave up**. A graph that opens downward is identified as begin **concave down**.

6. Compare the signs of the first and second differences for each function and its graph.

   a. How do the signs of the first differences for a linear function relate to the graph either increasing or decreasing?

   b. How do the signs of the second differences for quadratic functions relate to whether the parabola is opening upward or downward?

ACTIVITY 2.2

Analyzing the Leading Coefficient

You know that different forms of an equation can reveal different characteristics about functions. Quadratic functions can be written in different forms.

A quadratic function written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is in **standard form**. In this form, a and b are numerical coefficients and c is a constant.

A quadratic function written in **factored form** is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

Remember . . .
The leading coefficient of an equation is the numeric coefficient of the term with the greatest power.

1. Identify the standard form and factored form of each equation in the Getting Started.

2. Consider the leading coefficient of each function equation in both standard form and factored form.

   a. What does the leading coefficient tell you about the graph of each function?

   b. How is the leading coefficient related to the absolute minimum or absolute maximum of each function?

The graph of a quadratic function has either an absolute maximum or absolute minimum.

   c. How can you determine the y-intercept of the graph using standard form?

3. Determine from the equation whether each quadratic function has an absolute maximum or absolute minimum. Explain how you know.

a. $f(n) = 2n^2 + 3n - 1$

b. $g(x) = -2x^2 - 3x + 1$

c. $r(x) = -\frac{1}{2}x^2 - 3x + 1$

d. $b(x) = -0.009(x + 50)(x - 250)$

e. $f(t) = \frac{1}{3}(x - 1)(x - 1)$

f. $j(x) = -2x(1 - x)$

ACTIVITY 2.3 Axis of Symmetry

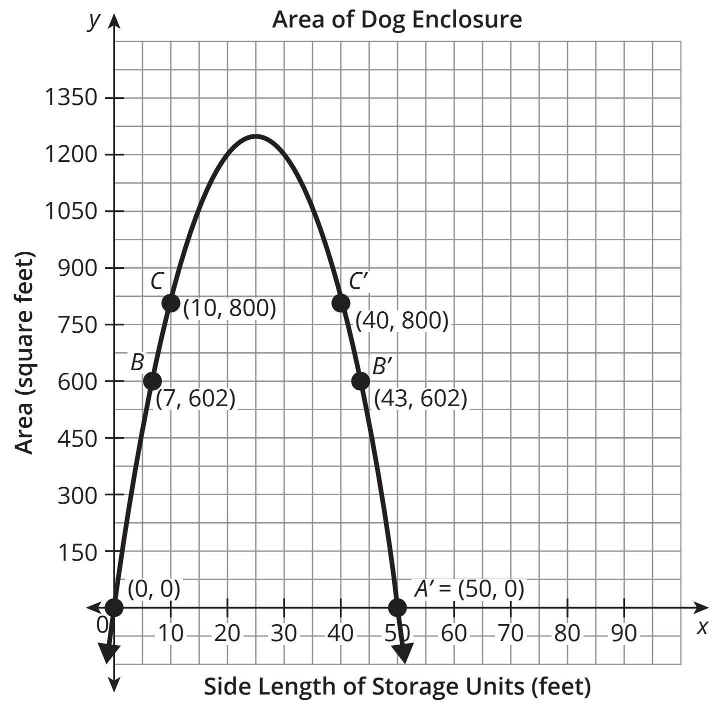
The **vertex of a parabola** is the lowest or highest point on the graph of the quadratic function. The **axis of symmetry** or the line of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images. Because the axis of symmetry always divides the parabola into two mirror images, you can say that a parabola has reflectional symmetry.

The vertex is identified as either the absolute minimum or absolute maximum.

1. Use patty paper to trace the graph representing the area of the dog enclosure. Then, fold the graph to show the symmetry of the parabola and trace the axis of symmetry.

   a. Place the patty paper over the original graph. What is the equation of the axis of symmetry?

   b. Draw and label the axis of symmetry on the graph from your patty paper.



2. Analyze the symmetric points labeled on the graph.

   a. What do you notice about the y-coordinates of the points?

   b. What do you notice about each point's horizontal distance from the axis of symmetry?

   c. How does the x-coordinate of each symmetric point compare to the x-coordinate of the vertex?

For a function in factored form, $f(x) = a(x - r_1)(x - r_2)$, the equation for the axis of symmetry is given by $x = \frac{r_1 + r_2}{2}$. For a quadratic function in standard form, $f(x) = ax^2 + bx + c$, the equation for the axis of symmetry is $x = \frac{-b}{2a}$.

3. Identify the axis of symmetry of the graph of each situation from the Getting Started using the factored form of each equation.

4. Describe the meaning of the axis of symmetry in each situation, when possible.

5. Describe how you can use the axis of symmetry to determine the ordered pair location of the absolute maximum or absolute minimum of a quadratic function given the equation for the function in factored form.

As you analyze a parabola from left to right, it will have either an interval of increase followed by an interval of decrease, or an interval of decrease followed by an interval of increase.

6. How does the absolute maximum or absolute minimum help you determine each interval?

Consider the graph of the quadratic function representing the Pumpkin Launch problem situation.

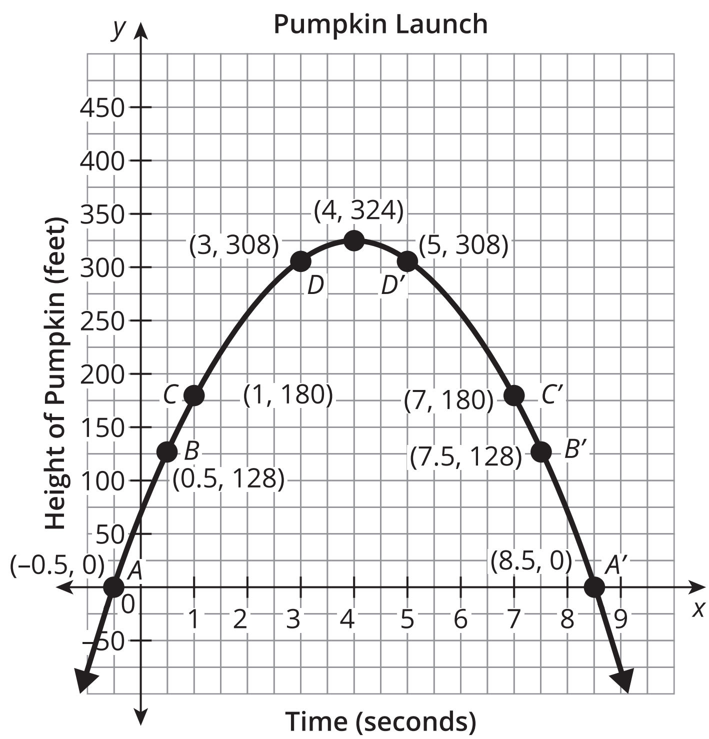
7. Determine the average rate of change between each pair. Then, summarize what you notice.

   a. Points *A* and *B*

   b. Points *A*′ and *B*′

   c. Points *B* and *C*

   d. Points *B*′ and *C*′

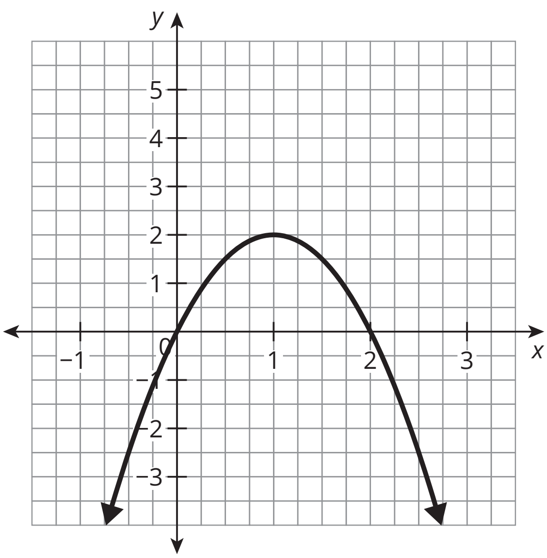


   e. What do you notice about the average rates of change between pairs of symmetric points?

The formula for the average rate of change is $\frac{f(b) - f(a)}{b - a}$.

8. For each function shown, identify the domain, range, *x*-intercepts, *y*-intercept, axis of symmetry, vertex, and interval of increase and decrease.

a. The graph shown represents the function $f(x) = -2x^2 + 4x$.



Domain:

Range:

***x*-intercepts:**

***y*-intercept:**

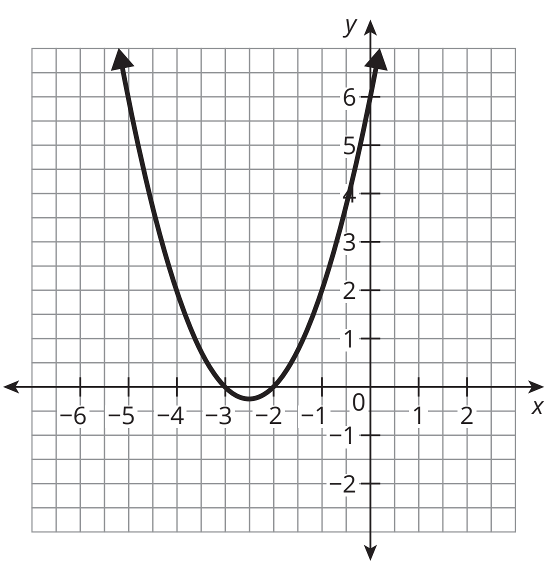
Axis of symmetry:

Vertex:

Interval of increase:

Interval of decrease:

b. The graph shown represents the function $f(x) = x^2 + 5x + 6$.



Domain:

Range:

***x*-intercepts:**

***y*-intercept:**

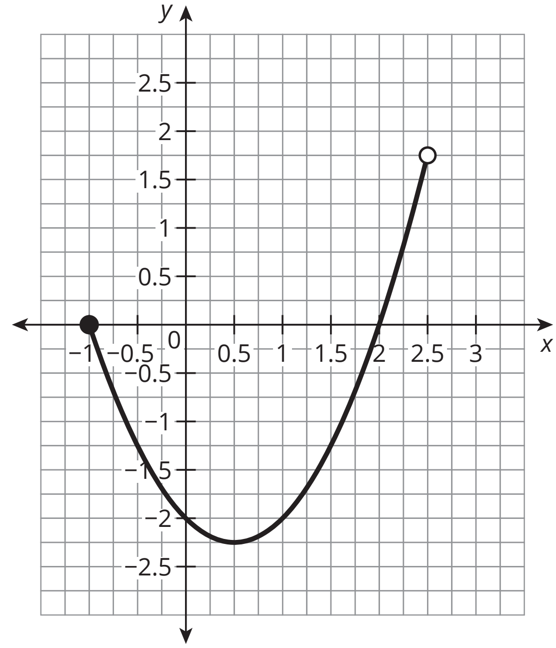
Axis of symmetry:

Vertex:

Interval of increase:

Interval of decrease:

c. The graph shown represents the function $f(x) = x^2 - x - 2$.



Domain: **Range:**

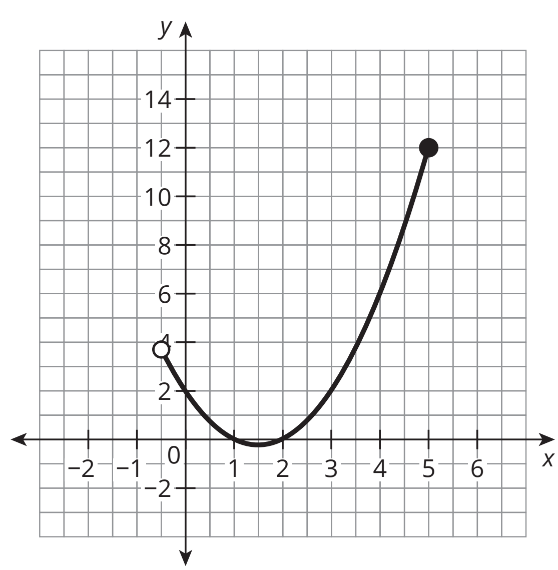
***x*-intercepts:** ***y*-intercept:**

Axis of symmetry: **Vertex:**

Interval of increase: **Interval of decrease:**

Ask Yourself . . .
How does an open or closed circle on a graph reflect the domain? The range?

d. The graph shown represents part of the function $f(x) = x^2 - 3x + 2$.



Domain: **Range:**

x-intercepts: **y-intercept:**

Axis of symmetry: **Vertex:**

Interval of increase: **Interval of decrease:**

ACTIVITY 2.4

Exploring Factored Form

When given a function $g(x)$ has a zero at $x = 4$, then $g(4) = 0$. This can also be interpreted as an x-intercept at (4, 0).

You have analyzed quadratic functions and their equations. Let's look at the factored form of a quadratic function in more detail.

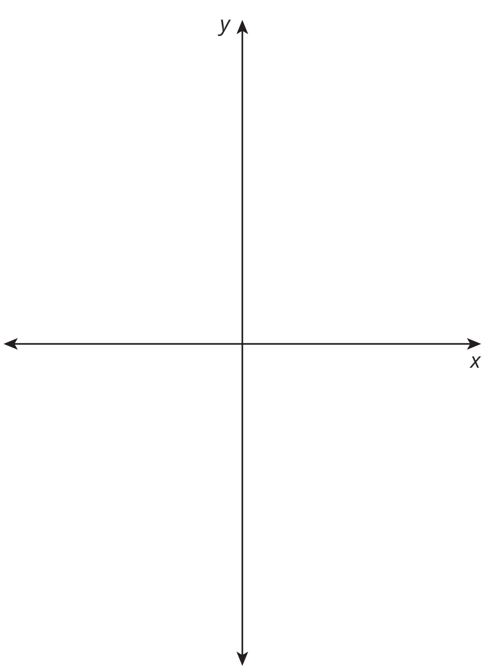
1. A group of students each write a quadratic function in factored form to represent a parabola that opens downward and has zeros at $x = 4$ and $x = -1$.

Gabriela
My function is
$k(x) = -(x - 4)(x + 1)$.

Alexander
My function is
$g(x) = -2(x - 4)(x + 1)$.

Fernando
My function is
$m(x) = 2(x - 4)(x + 1)$.

Michael
My function is
$f(x) = -(x + 4)(x - 1)$.



a. Sketch a graph of each student's function and label key points. What are the similarities among all the graphs? What are the differences among the graphs?

b. What would you tell Fernando and Michael to correct their functions?

c. How is it possible to have more than one correct function?

d. How many possible functions can represent the given characteristics? Explain your reasoning.

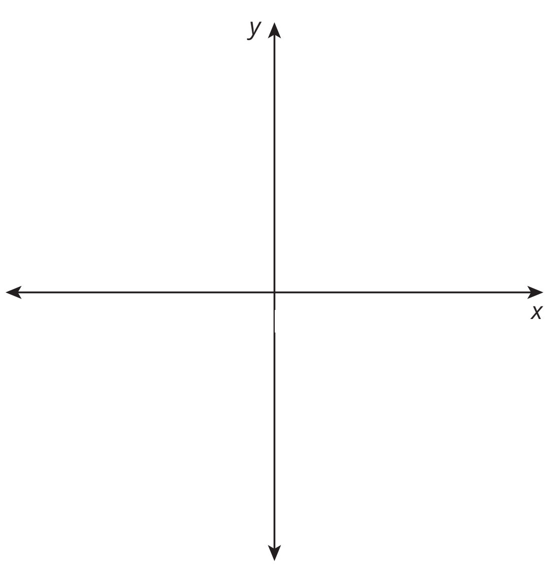
2. Consider a quadratic function written in factored form, $f(x) = a(x - r_1)(x - r_2)$.

a. What does the sign of the a-value tell you about the graph?

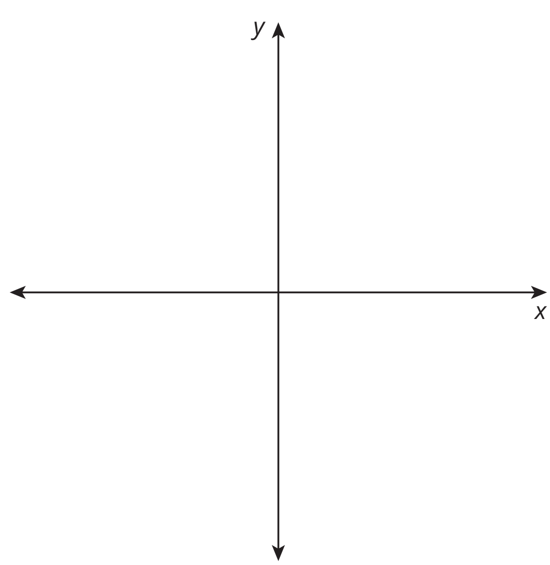
b. What do r_1 and r_2 tell you about the graph?

3. Use the given information to write a function in factored form. Sketch a graph of each function and label key points, which include the vertex and the x- and y-intercepts.

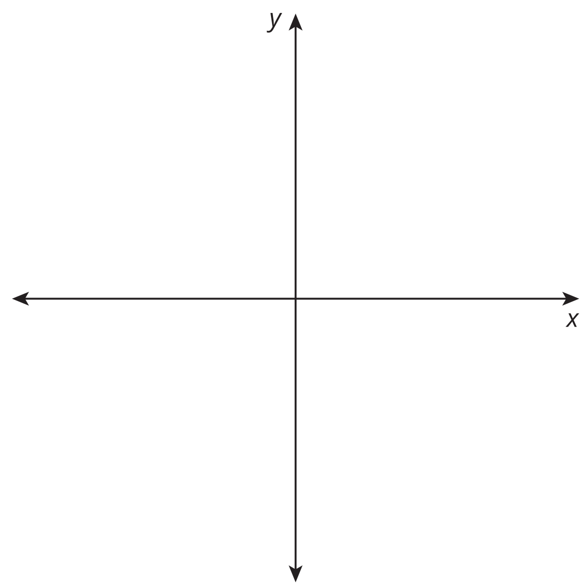
a. The parabola opens upward, and the zeros are at $x = 2$ and $x = 4$.



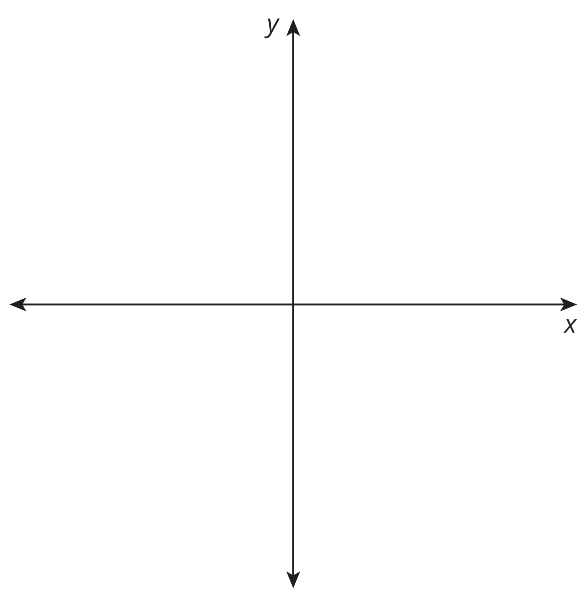
b. The parabola opens downward, and the zeros are at $x = -3$ and $x = 1$.



c. The parabola opens downward, and the zeros are at $x = 0$ and $x = 5$.



d. The parabola opens upward, and the zeros are at $x = -2.5$ and $x = 4.3$.



4. Compare your quadratic functions with your classmates' functions. How does the *a*-value affect the shape of the graph?

5. For each quadratic function,

- Use the standard form to determine the axis of symmetry, the absolute maximum or absolute minimum, and the y-intercept. Graph and label each characteristic.
- Use technology to identify the zeros. Label the zeros on the graph.
- Draw the parabola. Use the curve to write the function in factored form.
- Verify the function you wrote in factored form is equivalent to the given function in standard form.

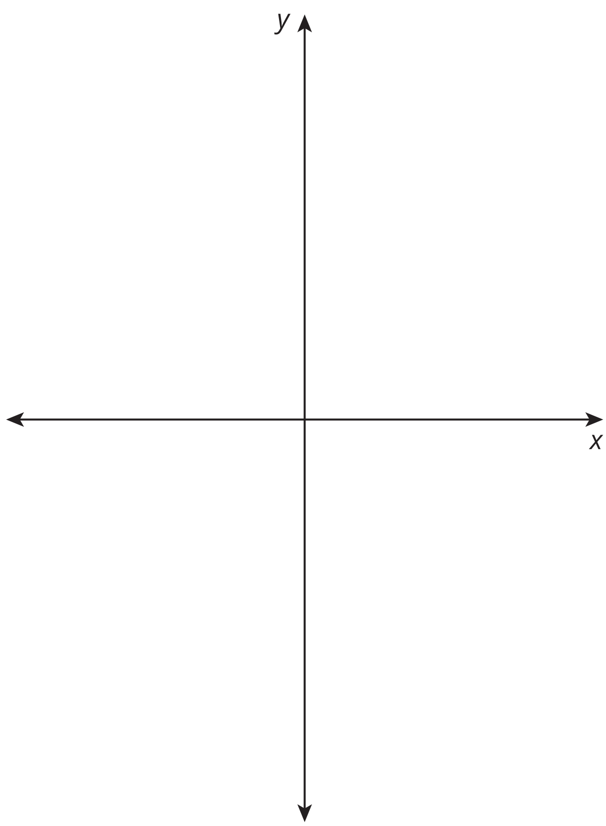
Remember . . .

A function written in standard form $f(x) = ax^2 + bx + c$ has an axis of symmetry at $x = \frac{-b}{2a}$.

a. $h(x) = x^2 - 8x + 12$

Zeros: ____________________

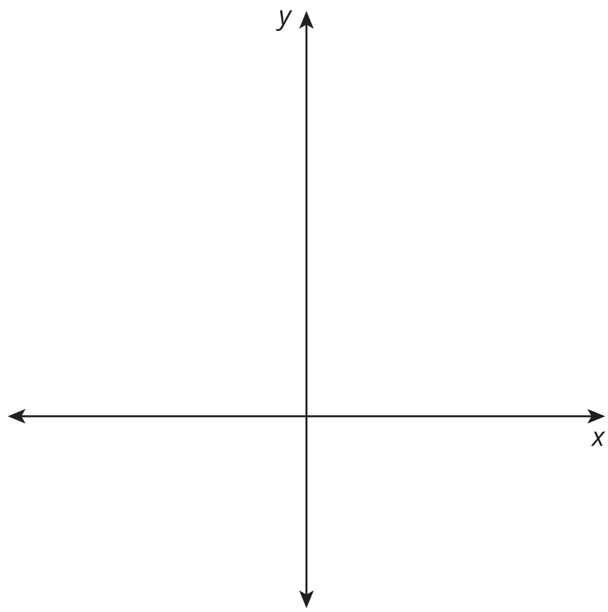
Factored form: ____________________



b. $r(x) = -2x^2 + 6x + 20$

Zeros: ____________________

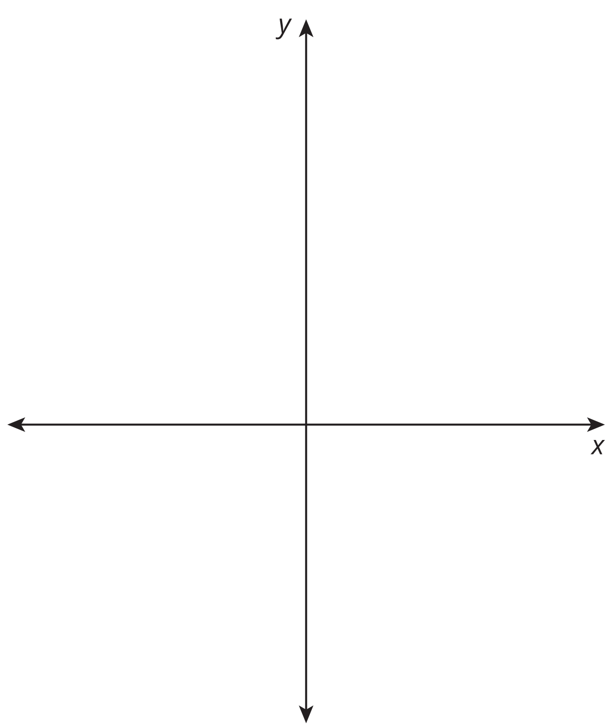
Factored form: ____________________



c. $w(x) = -x^2 - 4x$

Zeros: ____________________

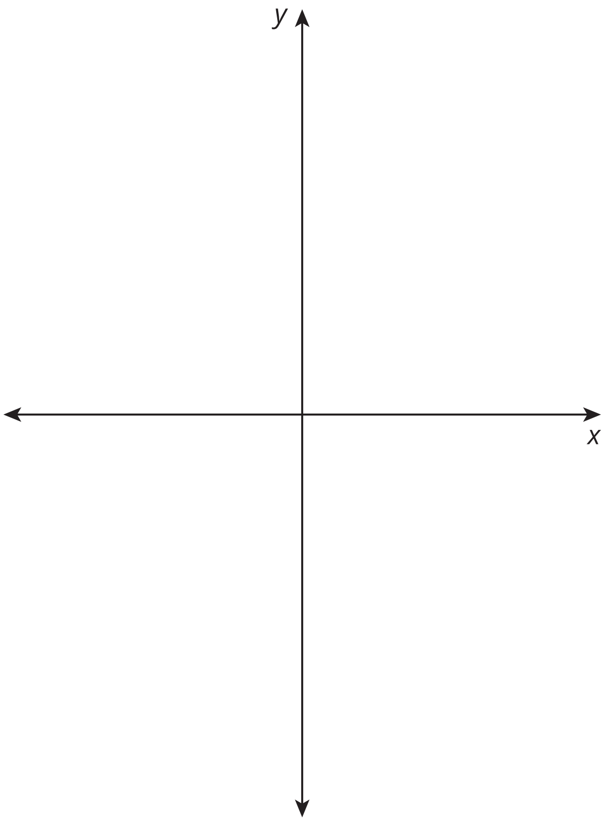
Factored form: ____________________



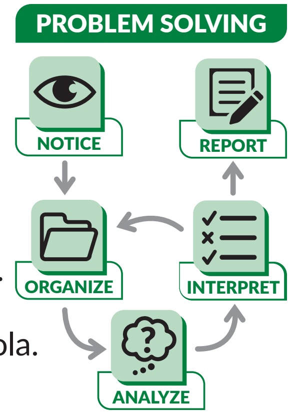
d. $c(x) = 3x^2 - 3$

Zeros: ______________________

Factored form: ______________________



Talk the Talk



Quadratic Sleuthing

Use the given information to answer each question. Do not use technology. Show your work.

Think About . . .
Sketch a graph by hand when you need a model.

1. Determine the axis of symmetry of each parabola.

   a. The x-intercepts of the parabola are (1, 0) and (5, 0).

   b. The x-intercepts of the parabola are (−3.5, 0) and (4.1, 0).

   c. Two symmetric points on the parabola are (−7, 2) and (0, 2).

2. Describe how to determine the axis of symmetry given the x-intercepts of a parabola.

3. Determine the location of the vertex of each parabola.

   a. The function $f(x) = x^2 + 4x + 3$ has the axis of symmetry of $x = -2$.

   b. The equation of the parabola is $y = x^2 - 4$, and the x-intercepts are (−2, 0) and (2, 0).

   c. The function $f(x) = x^2 + 6x - 5$ has two symmetric points, (−1, −10) and (−5, −10).

4. Describe how to determine the vertex of a parabola given the equation and the axis of symmetry.

5. Determine another point on each parabola.

   a. The axis of symmetry is $x = 2$, and a point on the parabola is (0, 5).

   b. The vertex is (0.5, 9), and an x-intercept is (−2.5, 0).

   c. The vertex is (−2, −8), and a point on the parabola is (−1, −7).

6. Describe how to determine another point on a parabola when you are given one point and the axis of symmetry.

Ask Yourself . . .
What precise mathematical language do you need to communicate your mathematical reasoning?

Lesson 2 Assignment

Write

1. Describe the characteristics of a quadratic function that you can determine from its equation in standard form.
2. Describe the characteristics of a quadratic function that you can determine from its equation in factored form.

Remember

The sign of the leading coefficient of a quadratic function in standard form or factored form describes whether the function has an absolute maximum or absolute minimum.

A parabola is a smooth curve with reflectional symmetry. The axis of symmetry contains the vertex of the graph of the function, which is located at the absolute minimum or absolute maximum of the function.

Practice

1. Analyze each quadratic function.

   $g(x) = 12x - 4x^2 + 16$ $\qquad$ $h(x) = \frac{1}{4}(x - 3)(x + 2)$

   a. Identify the quadratic function as standard form or factored form.

   b. Does the quadratic function have an absolute maximum or absolute minimum?

   c. Does the graph open upward or downward?

   d. Determine any intercepts from the given form of the function.

Lesson 2 Assignment

2. Analyze each quadratic function.

$f(x) = -\frac{2}{3}x^2 - 3x + 15$ $g(x) = \frac{3}{4}x^2 + 12x - 27$

a. Identify the axis of symmetry.

b. Use the axis of symmetry to determine the ordered pair of the absolute maximum or absolute minimum value.

c. Describe the intervals of increase and decrease.

d. Sketch the graph based on the information you just calculated.

Lesson 2 Assignment

e. Use technology to identify the zeros.

f. Place two pairs of symmetric points on your graph. What is the average rate of change between these pairs of symmetric points?

g. Write the function in factored form.

3. A parabola opens downward and has zeros at $x = -2$ and $x = 3$.

   a. Represent it as a quadratic equation in factored form.

   b. Sketch a graph of the quadratic function.

   c. What is the axis of symmetry and y-intercept of the quadratic function?

Lesson 2 Assignment

Prepare

1. Describe the effect of changing the a-value of the function $a \cdot f(x)$ given the parent function $f(x) = x$.

2. Describe the effect of changing the d-value of the function $f(x) + d$ given the parent function $f(x) = x$.

3 Quadratic Function Transformations

OBJECTIVES

- Experiment with transformations of quadratic functions using technology.
- Graph quadratic functions and transformations of quadratic functions.
- Determine the effect of replacing the parent quadratic function $f(x) = x^2$ with $f(x) + d$, $af(x)$ and $f(x - c)$ for different values of a, c, and d.
- Distinguish between function transformations that occur outside the function and inside the argument of the function.

NEW KEY TERMS

- argument of a function
- reflection
- line of reflection

You know how to transform linear functions.

How can you define quadratic functions and show transformations of this function type?

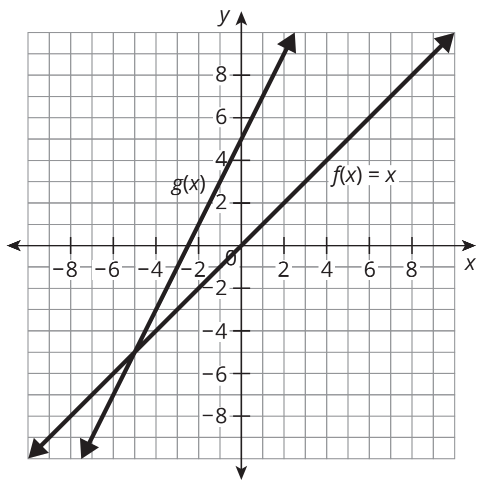
Getting Started

A Blast From the Past

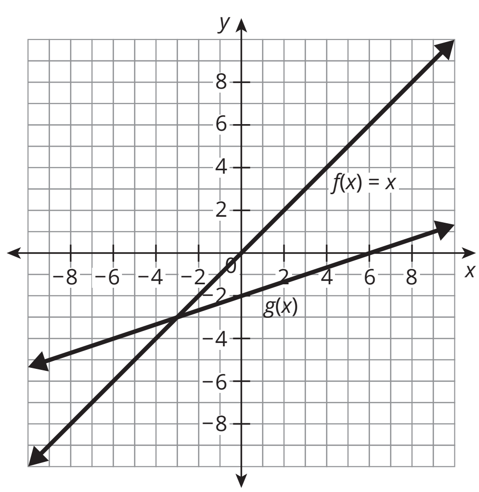
Let $f(x) = x$ and $g(x) = a \cdot f(x) + d$.

For each function and graph on the coordinate plane, describe the transformations.

1. $g(x) = 2(x) + 5$



2. $g(x) = \frac{1}{3}(x) - 2$



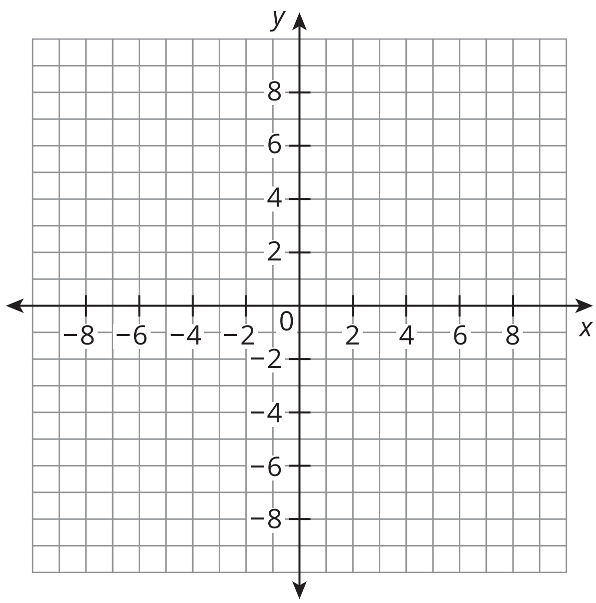
ACTIVITY 3.1

Transformations Outside the Function

Consider the three quadratic functions shown.

$g(x) = x^2$ $c(x) = x^2 + 3$ $d(x) = x^2 - 3$

1. Use technology to graph each function. Then, sketch and label the graph of each function.



2. Write the functions $c(x)$ and $d(x)$ in terms of the parent function $g(x)$. Then, describe the transformations of each function.

3. Describe the similarities and differences between the three graphs. How do these similarities and differences relate to the equations of the functions $g(x)$, $c(x)$ and $d(x)$?

Recall that the function $t(x)$ of the form $t(x) = f(x) + d$ is a vertical translation of the function $f(x)$. The value $|d|$ describes how many units up or down the graph of the original function is translated.

4. Describe each graph in relation to the parent function, $g(x) = x^2$. Then, use coordinate notation to represent the vertical translation.

   a. $f(x) = g(x) + d$, when $d > 0$

   b. $f(x) = g(x) + d$, when $d < 0$

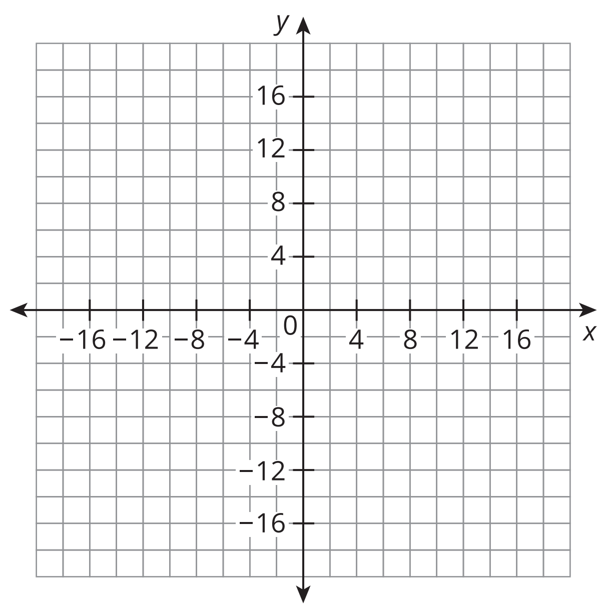
   c. Each point (x, y) on the graph of $g(x)$ becomes the point ______________ on $f(x)$.

Consider these quadratic functions.

$g(x) = x^2 \quad k(x) = \frac{1}{2}x^2$

$j(x) = 2x^2 \quad p(x) = -x^2$

5. Use technology to graph each function. Then, sketch and label the graph of each function.



6. Write the functions $j(x)$, $k(x)$, and $p(x)$ in terms of the parent function $g(x)$. Then, describe the transformations of each function.

Recall that a function $t(x)$ of the form $t(x) = a \cdot f(x)$ is a vertical dilation of the function $f(x)$. The a-value describes the vertical dilation of the graph of the original function.

7. Describe each graph in relation to the parent function $g(x) = x^2$. Then, use coordinate notation to represent the vertical dilation.

   a. $f(x) = a \cdot g(x)$, when $a > 1$

   b. $f(x) = a \cdot g(x)$, when $a < 0$

   c. $f(x) = a \cdot g(x)$, when $0 < a < 1$

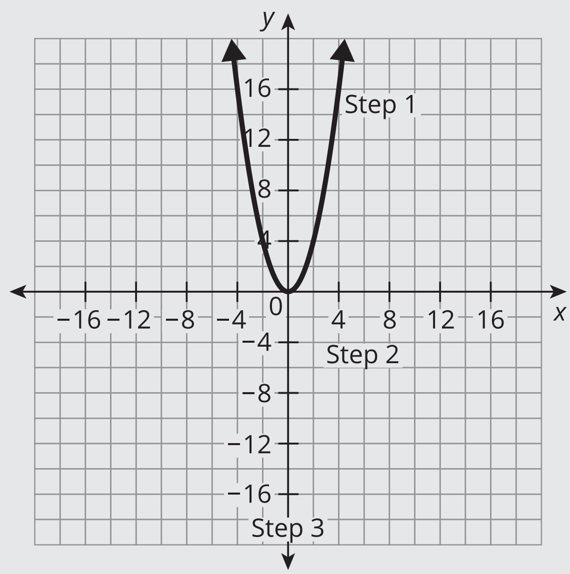
   d. Each point (x, y) on the graph of $g(x)$ becomes the point ______________ on $f(x)$.

You know that changing the *a*-value of a function to its opposite reflects the function across a horizontal line. But, the line of reflection for the function might be different, depending on how you write the transformation and the order in which the transformations are applied.

8. Jacob and Hannah each sketched a graph of the function $b(x) = -(x^2) - 3$ using different strategies. Write the step-by-step reasoning that each student used.

A *reflection of a graph* is the mirror image of the graph about a line of reflection.

Jacob



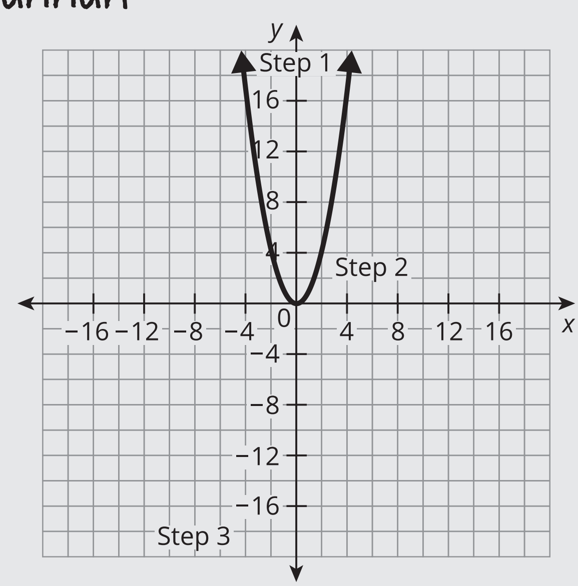
Step 1:

Step 2:

Step 3:

A *line of reflection* is the line that the graph is reflected across. A horizontal line of reflection affects the *y*-coordinates.

Hannah



Step 1:

Step 2:

Step 3:

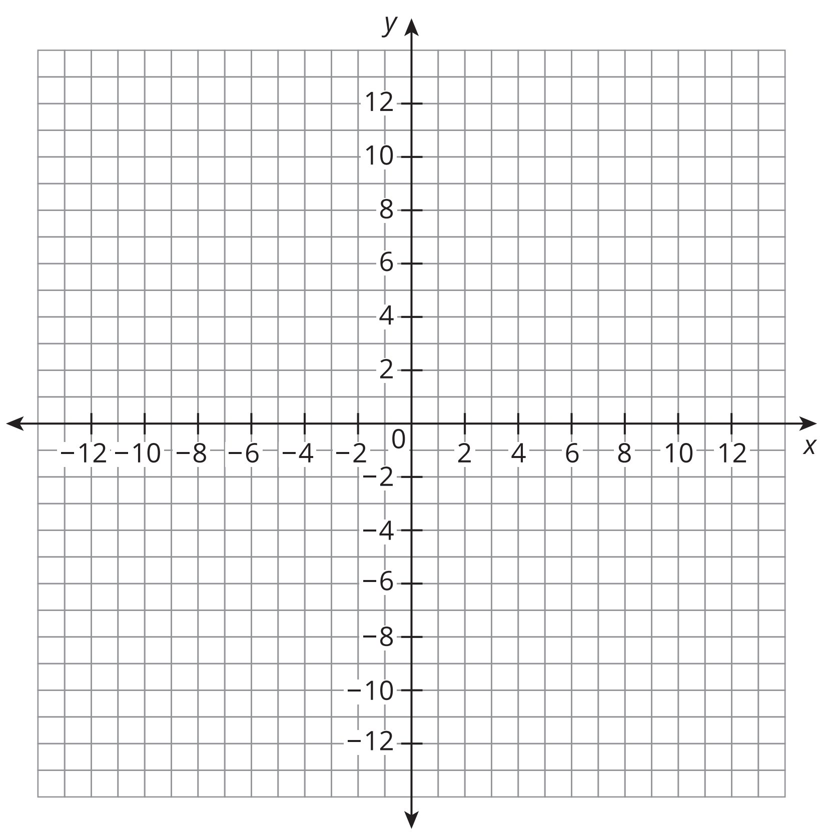
9. Explain how changing the order of the transformations affects the line of reflection.

Given the function $f(x) = x^2$, use the coordinate plane below to answer Questions 10 through 14.

10. Consider the function $a(x) = 2f(x) + 1$.

    a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $a(x)$.

    b. Graph and label $a(x)$ on the coordinate plane below.



11. Consider the function $b(x) = -2f(x) + 1$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $b(x)$.

b. Graph and label $b(x)$ on the coordinate plane above.

12. Describe the graph of $b(x)$ in terms of $a(x)$.

13. Consider the function $-a(x)$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $a(x)$ becomes a point on the graph of $-a(x)$.

b. Graph and label $-a(x)$ on the coordinate plane on the previous page.

14. Describe the graph of $-a(x)$ in terms of $a(x)$.

ACTIVITY 3.2 Horizontal Translations of Quadratic Functions

Consider the three quadratic functions shown, where $h(x) = x^2$ is the parent function. The operations are performed on x, which is the argument of the function.

- $h(x) = x^2$
- $v(x) = (x + 3)^2$
- $w(x) = (x - 3)^2$

The **argument of a function** is the expression inside the parentheses. For $y = f(x - c)$ the expression $x - c$ is the argument of the function.

You can write the given functions $v(x)$ and $w(x)$ in terms of the parent function $h(x)$.

WORKED EXAMPLE

To write $v(x)$ in terms of $h(x)$, you just substitute $x + 3$ into the argument for $h(x)$, as shown.

$$h(x) = x^2$$

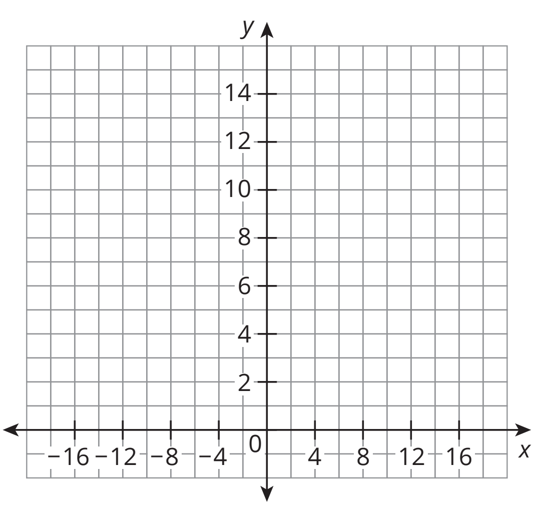
$$v(x) = h(x + 3) = (x + 3)^2$$

So, $x + 3$ replaces the variable x in the function $h(x) = x^2$.

1. Write the function $w(x)$ in terms of the parent function $h(x)$.

Sketch the graphs one at a time to help you see which is which.

2. Sketch and label the graph of each function. Identify key points.



3. Compare the graphs of $v(x)$ and $w(x)$ to the graph of the parent function. What do you notice?

Think About . . .
Notice there are no negative y-values in this table. Are negative values included in the range of h, v, or w?

4. Write the x-value of each of the corresponding reference points on $v(x)$ and $w(x)$.

$h(x) = x^2$	$v(x) = (x + 3)^2$	$w(x) = (x - 3)^2$
(−2, 4)	(____, 4)	(____, 4)
(−1, 1)	(____, 1)	(____, 1)
(0, 0)	(____, 0)	(____, 0)
(1, 1)	(____, 1)	(____, 1)
(2, 4)	(____, 4)	(____, 4)

5. Use the table to compare the ordered pairs of the graphs of $v(x)$ and $w(x)$ to the ordered pairs of the graph of the parent function $h(x)$. What do you notice?

6. Complete each sentence with the coordinate notation to represent the horizontal translation of each function.

   a. $v(x) = h(x + 3)$
   Each point (x, y) on the graph of $h(x)$ becomes the point _______________ on $v(x)$.

   b. $w(x) = h(x - 3)$
   Each point (x, y) on the graph of $h(x)$ becomes the point _______________ on $w(x)$.

When a constant is added or subtracted outside a function, like $g(x) + 3$ or $g(x) - 3$, then only the y-values change, resulting in a vertical translation. When a constant is added or subtracted inside a function, like $g(x + 3)$ or $g(x - 3)$, then only the x-values change, resulting in a horizontal translation.

7. Describe each graph in relation to the parent function $h(x) = x^2$.

   a. Compare $f(x) = h(x - c)$ to the parent function for $c > 0$.

   b. Compare $f(x) = h(x - c)$ to the parent function for $c < 0$.

Recall that for the parent function, the c-value of the transformed function $y = f(x - c)$ affects the input values of the function. The value $|c|$ describes the number of units the graph of $f(x)$ is translated right or left. When $c > 0$, the graph is translated to the right; when $c < 0$, the graph is translated to the left.

8. What generalization can you make about the effects of horizontal translations on the domain, range, and vertex of quadratic functions?

ACTIVITY 3.3

Reflections of Quadratic Functions

Consider the three quadratic functions below, where $h(x) = x^2$ is the parent function.

- $h(x) = x^2$
- $m(x) - (x^2)$
- $n(x) = (-x)^2$

1. Write the functions $m(x)$ and $n(x)$ in terms of the parent function $h(x)$.

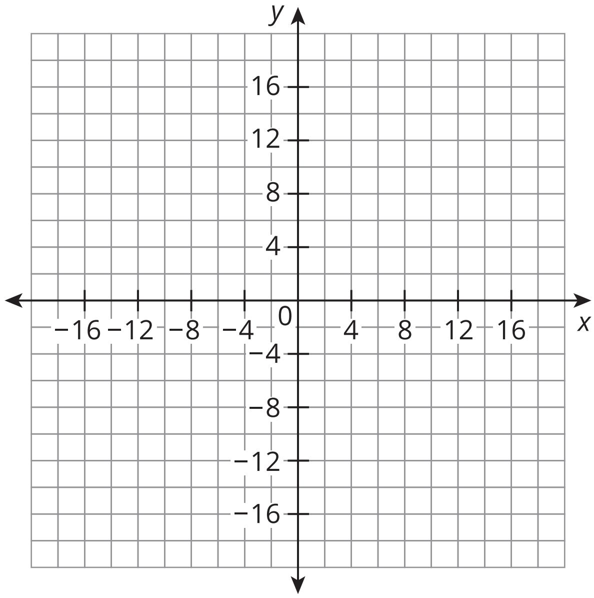
   a. $m(x) =$

   b. $n(x) =$

2. Compare $m(x)$ to $h(x)$. Does an operation performed on $h(x)$ or on the argument of $h(x)$ result in the equation for $m(x)$? What is the operation?

3. Compare $n(x)$ to $h(x)$. Does an operation performed on $h(x)$ or on the argument of $h(x)$ result in the equation for $n(x)$? What is the operation?

4. Use technology to sketch and label each function.



5. Compare the graphs of $m(x)$ and $n(x)$ to the graph of the parent function $h(x)$. What do you notice?

6. Write the x- or y-value of each of the corresponding reference points on $m(x)$ and $n(x)$.

$h(x) = x^2$	$m(x) = -(x^2)$	$n(x) = (-x)^2$
(−2, 4)	(−2, ____)	(____, 4)
(−1, 1)	(−1, ____)	(1, 1)
(0, 0)	(0, ____)	(____, 0)
(1, 1)	(1, ____)	(____, 1)
(2, 4)	(2, ____)	(____, 4)

7. Use the table to compare the ordered pairs of the graphs of $m(x)$ and $n(x)$ to the ordered pairs of the graph of the parent function $h(x)$. What do you notice?

When the negative is on the outside of the function, like $-g(x)$, all the y-values become the opposite of the y-values of $g(x)$. The x-values remain unchanged.

Remember, a **reflection** of a graph is a mirror image of the graph about a line of reflection. A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the y-coordinates, and a vertical line of reflection affects the x-coordinates.

8. Consider the graphs of $m(x)$ and $n(x)$.

   a. Which function represents a reflection of $h(x)$ across a horizontal line? Name the line of reflection.

When the negative is on the inside of the function, like $g(-x)$, all the x-values become the opposite of the x-values of $g(x)$. The y-values remain unchanged.

   b. Which function represents a reflection of $h(x)$ across a vertical line? Name the line of reflection.

9. Complete each sentence with the coordinate notation to represent the reflection of each function.

   a. $m(x) = -h(x)$
   Each point (x, y) on the graph of $h(x)$ becomes the point ______________ on $m(x)$.

   b. $n(x) = h(-x)$
   Each point (x, y) on the graph of $h(x)$ becomes the point ______________ on $n(x)$.

Talk the Talk

Write Them Up!

1. Consider the function, $f(x) = x^2$. Write the function in transformation function form in terms of the transformations described, then write an equivalent equation.

Transformation	Transformation Function Form	Equation
a. Reflection across the x-axis		
b. Horizontal translation of 2 units to the left and a vertical translation of 3 units up		
c. Vertical stretch of 2 units and a reflection across the line $y = 0$		
d. Vertical dilation of 2 units and a reflection across the line $y = 3$		
e. Horizontal translation of 3 units to the right, a vertical translation down 2 units, and a vertical dilation of $\frac{1}{2}$		
f. Vertical compression by a factor of 4		
g. Vertical stretch by a factor of 4		

Lesson 3 Assignment

Write

Given a parent function $y = f(x)$ and a function written in transformation form $g(x) = a \cdot f(x - c) + d$, describe how the transformations that are inside a function affect a graph differently than those on the outside of the function.

Remember

The parent quadratic function is $f(x) = x^2$.

The transformed function $y = f(x) + d$ shows a vertical translation of the function.

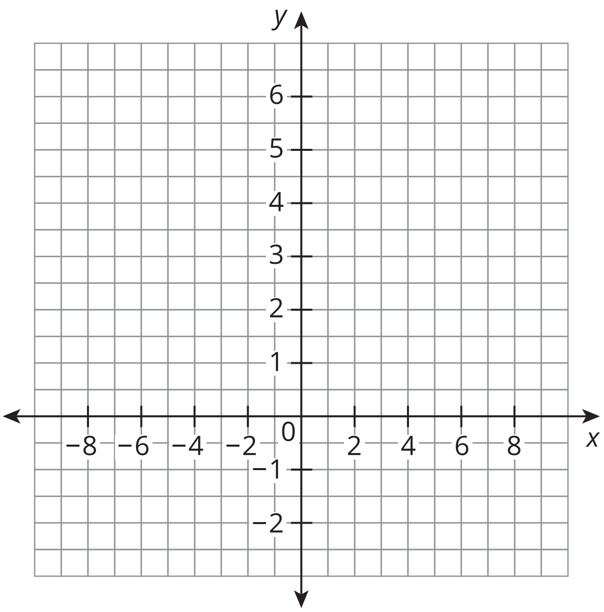
The transformed function $y = af(x)$ shows a vertical dilation of the function when $a > 0$, and when $a < 0$, it shows a vertical dilation and reflection across the x-axis.

The transformed function $y = f(x - c)$ shows a horizontal translation of the function.

Practice

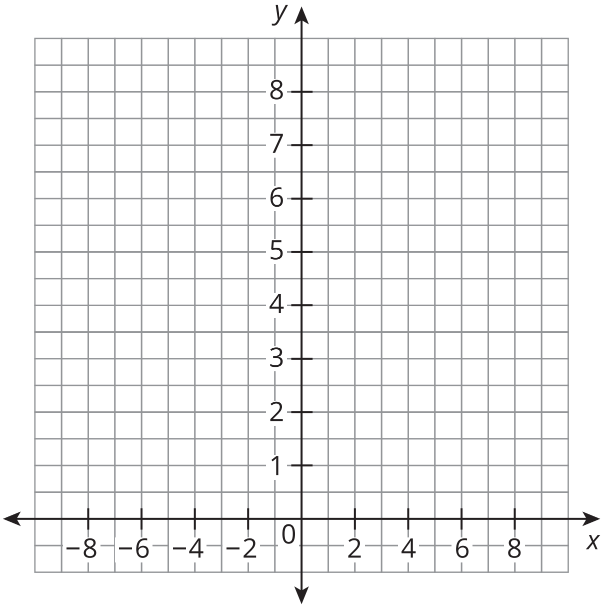
Given the parent function $f(x) = x^2$, consider each transformation. Describe how the transformations affected $f(x)$. Then, use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of the transformed function. Finally, sketch a graph of each new function.

1. $g(x) = \frac{1}{3} f(x) - 2$

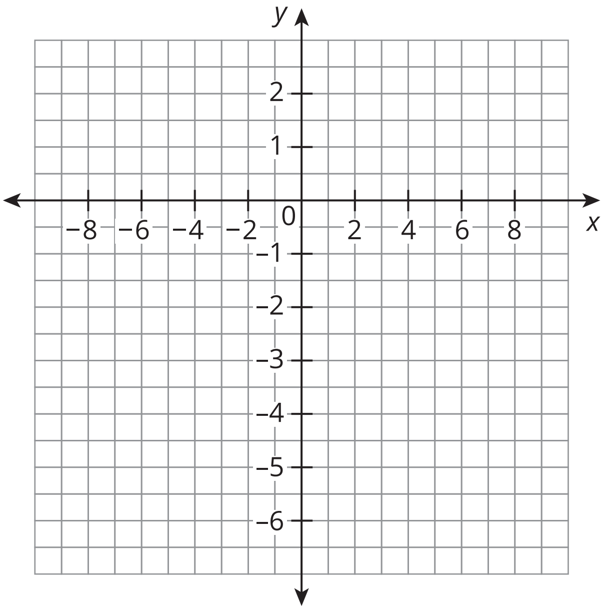


Lesson 3 Assignment

2. $j(x) = 2f(x + 1) + 4$

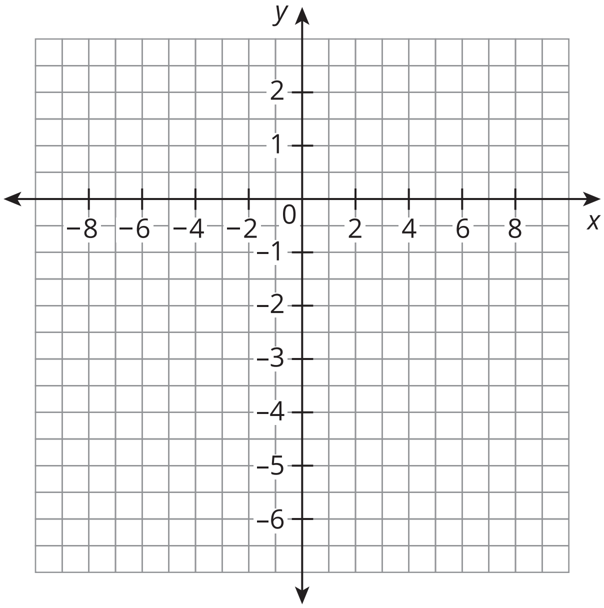


3. $m(x) = -\frac{1}{2}f(x - 3) - 1$



Lesson 3 Assignment

4. $p(x) = -f(x + 4) + 3$



Prepare

Write the equation for the axis of symmetry given each quadratic function.

1. $f(x) = -3x^2 - 4x + 5$

2. $f(x) = \frac{1}{4}(x - 1)(x + 2)$

3. $f(x) = -x^2 + 3$

4 Horizontal Transformations and Vertex Form

OBJECTIVES

- Dilate quadratic functions horizontally.
- Write equations of quadratic functions given multiple transformations.
- Graph quadratic functions given multiple transformations.
- Identify multiple transformations of quadratic functions given equations.
- Understand the form in which a quadratic function is written can reveal different key characteristics.
- Write quadratic equations in vertex and factored form.

NEW KEY TERM

- vertex form

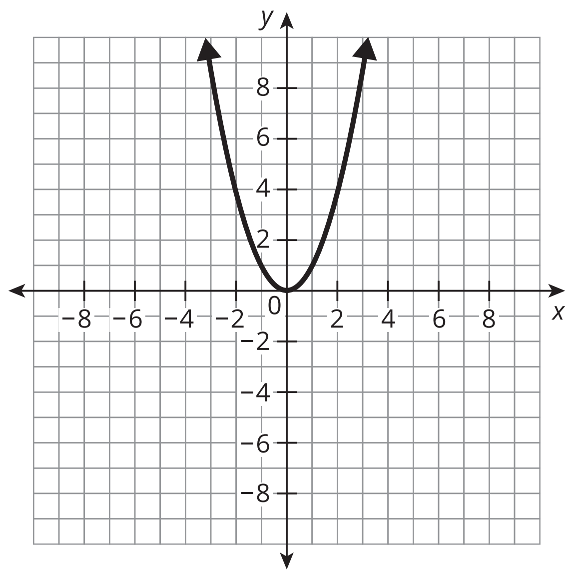
You know how to transform linear functions.

How can you apply what you know about the transformation form of a function, $g(x) = a \cdot f(b(x - c)) + d$, to quadratic functions?

Getting Started

H, I, J, . . .

Consider the function graphed.



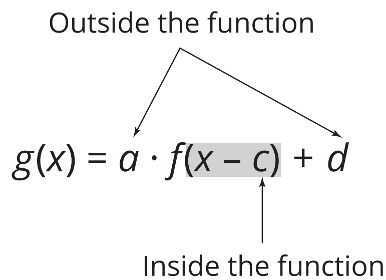
1. Identify each part of the graphed function.

   a. Domain

   b. Range

   c. Vertex

Recall that the transformation form of a function $y = f(x)$ can be written as shown.



2. How do the *a*-, *c*-, and *d*-values affect the graph of $f(x)$?

ACTIVITY 4.1

Horizontal Dilations of Quadratic Functions

In the previous lesson, you considered a transformation that affects the input values of a function. For the parent function, the *b*-value of the transformed function $y = f(bx)$ affects the input values of the function. For $b = -1$, the graph is reflected across the line $x = 0$, or the *y*-axis.

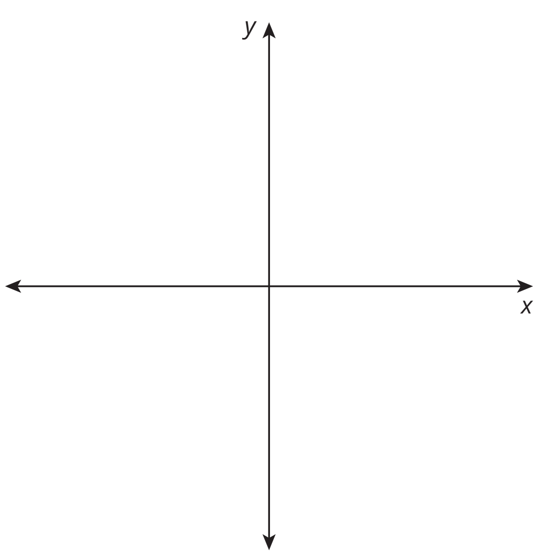
Let's consider different *b*-values and their effect on a parent function.

1. Consider the three quadratic functions, where $f(x) = x^2$ is the parent function.

   - $f(x) = x^2$
   - $t(x) = (3x)^2$
   - $q(x) = \left(\frac{1}{3}x\right)^2$

   a. Write the functions $t(x)$ and $q(x)$ in terms of the parent function $f(x)$. For each, determine whether you are performing an operation on the function $f(x)$ or on the argument of the function $f(x)$. Describe the operation.

   b. Sketch the graph of each function. Label each graph and include key points.



c. Use coordinate notation to represent the dilation of each function. Each point (x, y) on the graph of $f(x)$:

- becomes the point ______________________ on the graph of $t(x)$.
- becomes the point ______________________ on the graph of $q(x)$.

A horizontal dilation is a type of transformation that stretches or compresses the entire graph. Horizontal stretching is the stretching of a graph away from the y-axis. Horizontal compression is the squeezing of a graph toward the y-axis.

Consider the quadratic functions, where $h(x) = x^2$ is the parent function.

- $w(x) = \left(\frac{1}{4}x\right)^2$
- $z(x) = (4x)^2$

2. Write the functions $w(x)$ and $z(x)$ in terms of $h(x)$.

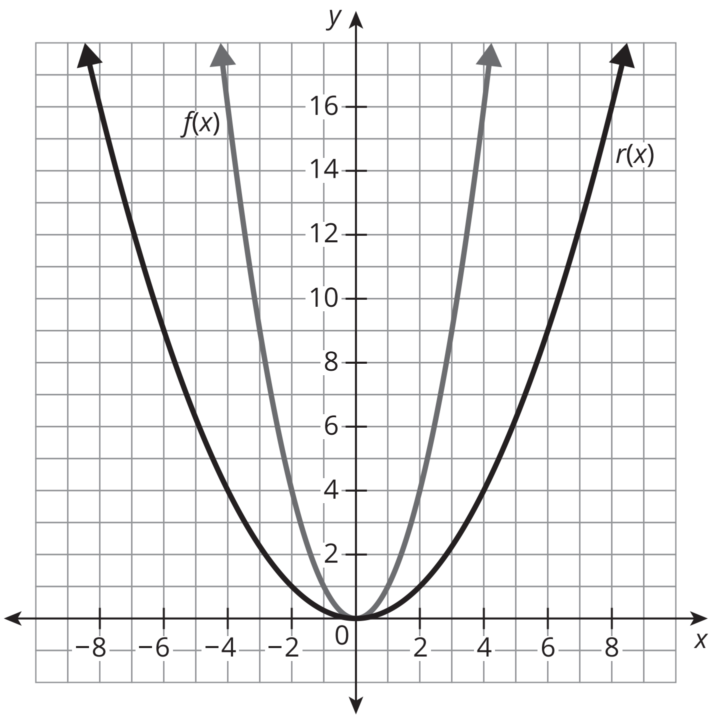
3. Write the x-value of each of the corresponding reference points on $w(x)$ and $z(x)$.

$h(x) = x^2$	$w(x) = \left(\frac{1}{4}x\right)^2$	$z(x) = (4x)^2$
(−2, 4)	(___, 4)	(___, 4)
(−1, 1)	(___, 1)	(___, 1)
(0, 0)	(___, 0)	(___, 0)
(1, 1)	(___, 1)	(___, 1)
(2, 4)	(___, 4)	(___, 4)

4. Use the table to compare the ordered pairs of the graphs of $w(x)$ and $z(x)$ to the ordered pairs of the graph of the parent function $h(x)$. What do you notice?

5. Given that the point (x, y) is on the graph of the function $y = f(x)$, what ordered pair describes a point on the graph of $g(x) = f(bx)$?

6. Now, let's compare the graph of $f(x) = x^2$ with $r(x) = f\left(\frac{1}{2}x\right)$.



a. Analyze the table of values that correspond to the graph.

Circle instances where the y-values for each function are the same. Then, list all the points where $f(x)$ and $r(x)$ have the same y-value. The first instance has been circled for you.

x	$f(x) = x^2$	$r(x) = f\left(\frac{1}{2}x\right)$
0	0	0
1	1	0.25
2	4	1
3	9	2.25
4	16	4
5	25	6.25
6	36	9

b. How do the x-values compare when the y-values are the same?

c. Complete the statement.
The function $r(x)$ is a ______________________ of $f(x)$ by a factor of _______.

d. How does the factor of stretching or compression compare to the b-value in $r(x)$?

Compared with the graph of $f(x)$, the graph of $f(bx)$ is:

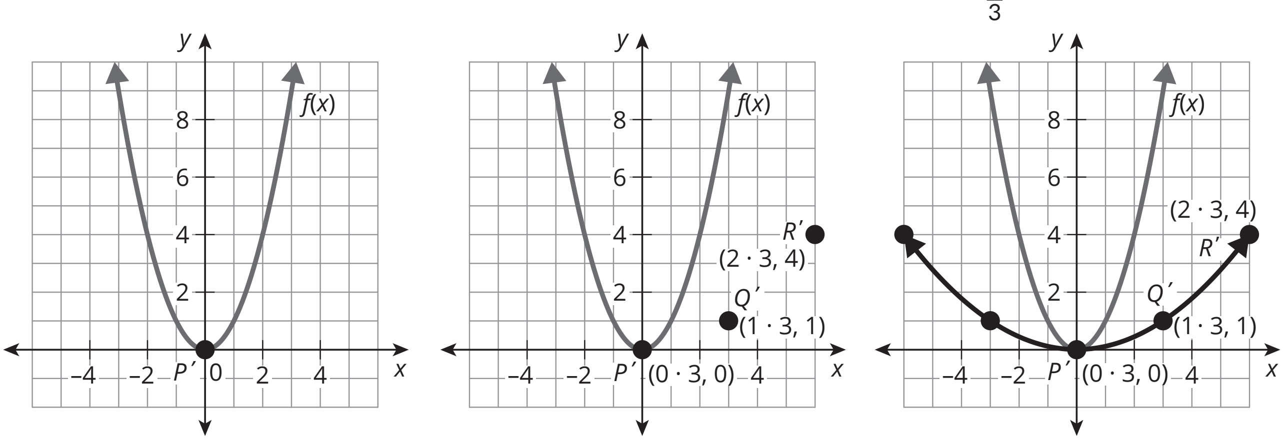
- horizontally compressed by a factor of $\frac{1}{|b|}$ when $|b| > 1$.
- horizontally stretched by a factor of $\frac{1}{|b|}$ when $0 < |b| < 1$.

WORKED EXAMPLE

You can use reference points to graph the function $q(x) = f\left(\frac{1}{3}x\right)$ when $f(x) = x^2$.

From $q(x)$, you know that $c = 0$, $d = 0$, and $b = \frac{1}{3}$. The vertex for $q(x)$ is (0, 0).

Notice $0 < |b| < 1$, so the graph will horizontally stretch by a factor of $\frac{1}{\frac{1}{3}}$, or 3.



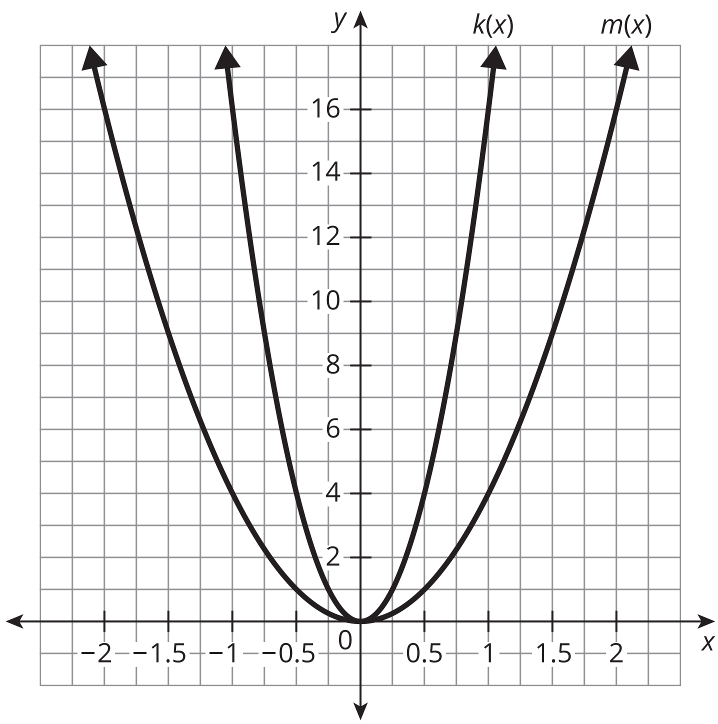
The function $f(x)$ is shown. First, plot the new vertex (c, d). This point establishes the new set of axes.

→ Next, think about b. To plot Q', move right $1 \cdot 3$ units and up 1 unit from the vertex because all x-coordinates are being stretched by a factor of 3. To plot R', start at the vertex, move to the right $2 \cdot 3$ units, and go up 4 units.

→ Finally, use symmetry to complete the graph.

7. Suppose you were going to graph $p(x) = f(3x)$. Describe how the graph would change. When (x, y) is any point on $f(x)$, describe any point on $p(x)$.

8. Consider the graph showing the quadratic functions $k(x)$ and $m(x)$. Amir and Madison are writing the function $m(x)$ in terms of $k(x)$.



Amir says that $m(x)$ is a transformation of the a-value.

$$m(x) = \frac{1}{4}k(x)$$

Madison says that $m(x)$ is a transformation of the b-value.

$$m(x) = k\left(\frac{1}{2}x\right)$$

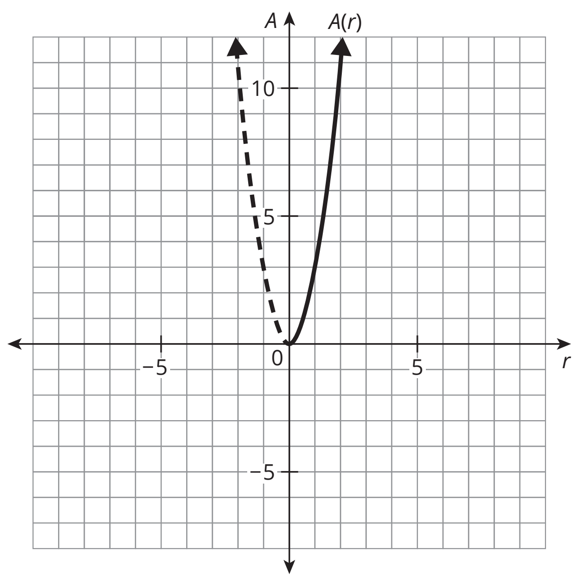
Who's correct? Justify your reasoning.

9. Describe how you can rewrite a quadratic function with a *b*-value transformation as a quadratic function with an *a*-value transformation.

10. Rewrite the function from the Worked Example, $q(x) = f\left(\frac{1}{3}x\right)$, without a *b*-value.

Consider the formula to calculate the area of a circle, $a = \pi r^2$. You can represent the area formula as the function $A(r) = \pi r^2$ and represent it on a coordinate plane.

Ask Yourself . . .
Why is part of the graph represented with a dashed smooth curve?



11. How would doubling the radius affect the area? Explain the change in area in terms of a transformation of the graph.

ACTIVITY **4.2**

Using Reference Points to Graph Quadratic Functions

> **Think About . . .**
> What is the pattern of the *a*-value when transforming the parent quadratic function?

Given $y = f(x)$ is the parent quadratic function, you can use reference points to graph $y = af(b(x - c)) + d$. Any point (x, y) on $f(x)$ maps to the point $\left(\frac{1}{b}x + c, ay + d\right)$.

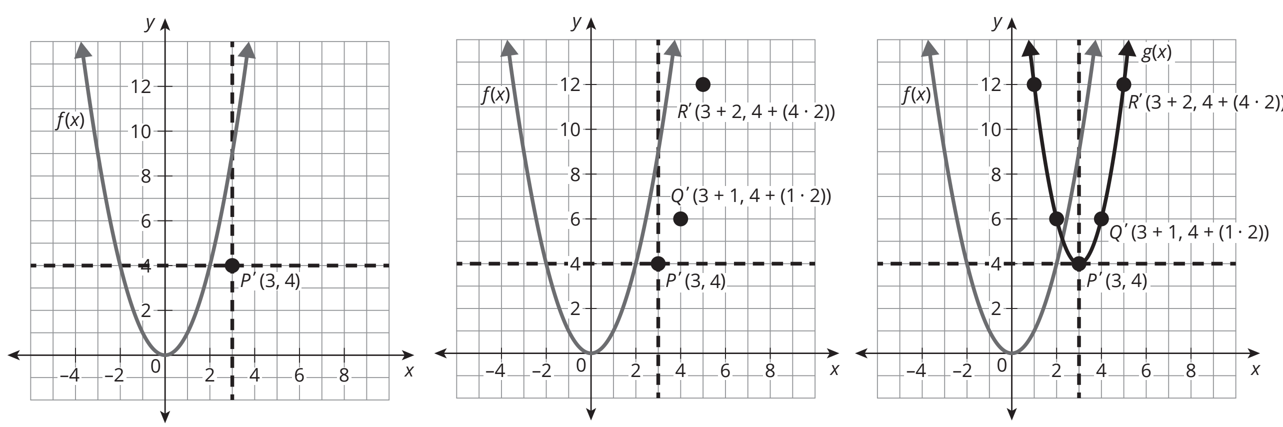
WORKED EXAMPLE

Given $f(x) = x^2$, graph the function $g(x) = 2f(x - 3) + 4$.

You can use reference points for $f(x)$ and your knowledge about transformations to graph the function $g(x)$.

From $g(x)$, you know that $a = 2$, $c = 3$, and $d = 4$.

The vertex for $g(x)$ will be at (3, 4). Notice $a > 0$, so the graph of the function will vertically stretch by a factor of 2.

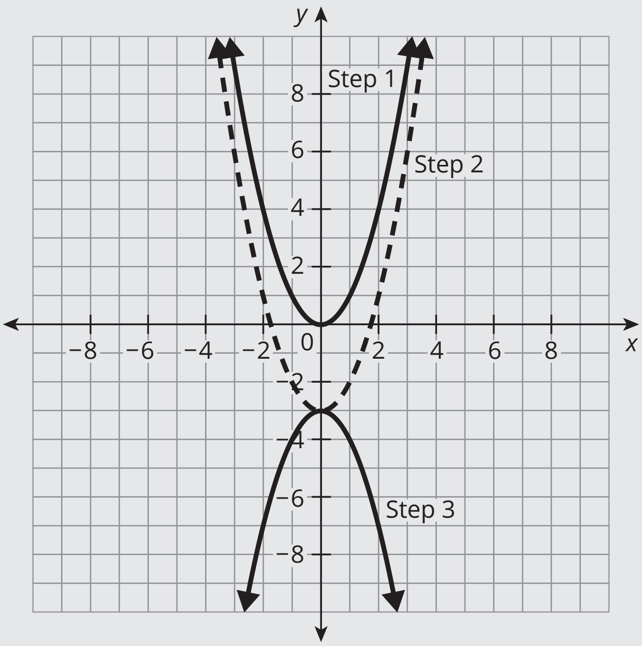


First, plot the new vertex, (*c*, *d*). This point establishes the new set of axes. →	Next, think about the reference points for the parent quadratic function and that $a = 2$. To plot point Q', move right 1 unit and up, not 1, but $1 \cdot 2$ units from the vertex P' because all *y*-coordinates are being multiplied by a factor of 2. To plot point R', move right 2 units from P' and up, not 4, but $4 \cdot 2$ units. →	Finally, use symmetry to complete the graph.

1. William, Maria, and Abby each sketched a graph of the equation $y = -x^2 - 3$ using different strategies. Provide the step-by-step reasoning that each student used.

William

a = -1 and d = -3



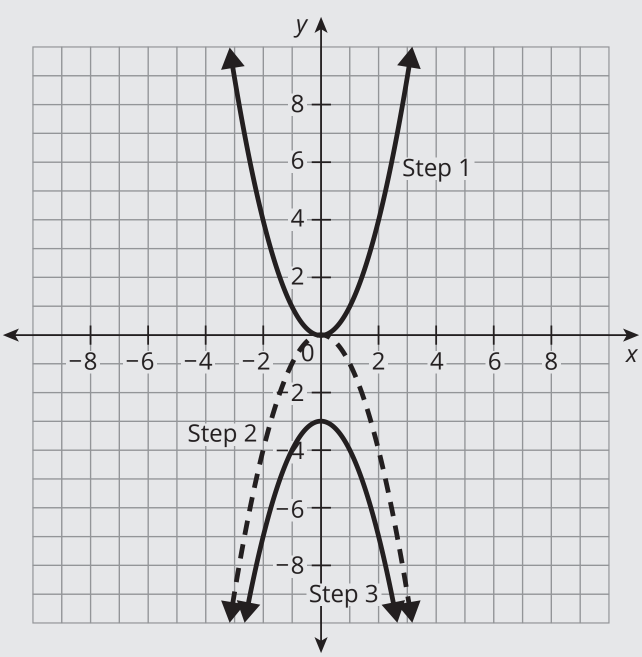
Step 1:

Step 2:

Step 3:

Maria

d = -3 and a = -1



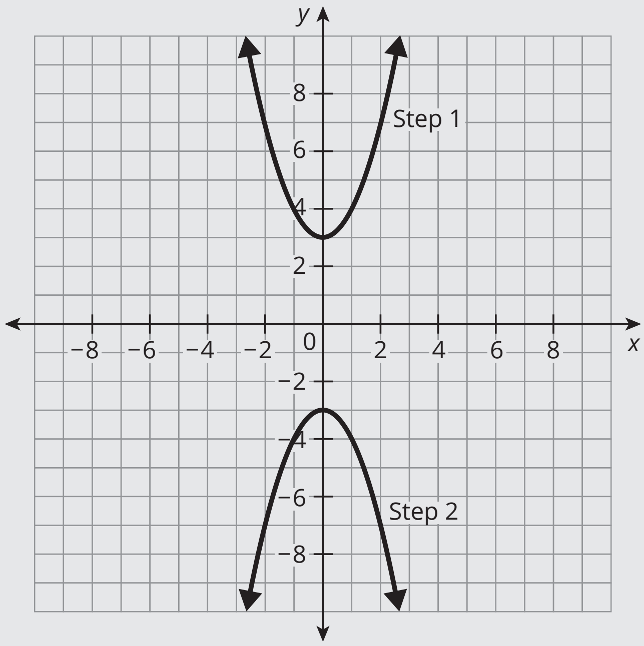
Step 1:

Step 2:

Step 3:

Abby

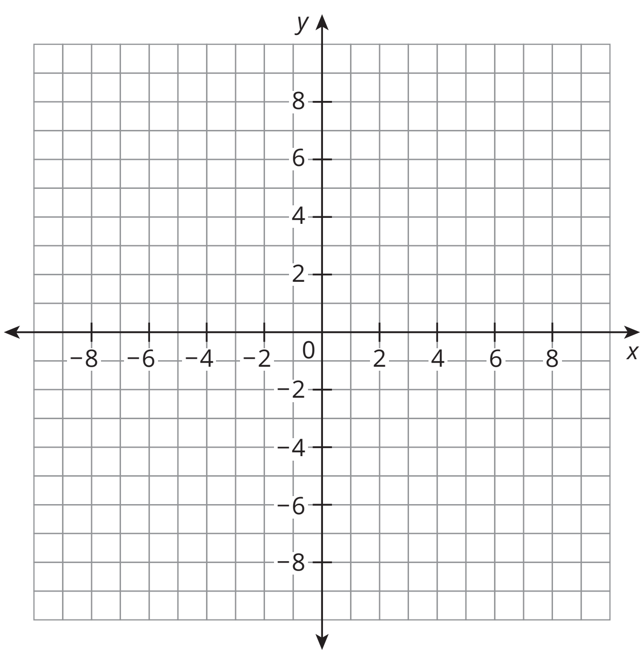
I rewrote the equation as $y = -(x^2 + 3)$.



Step 1:

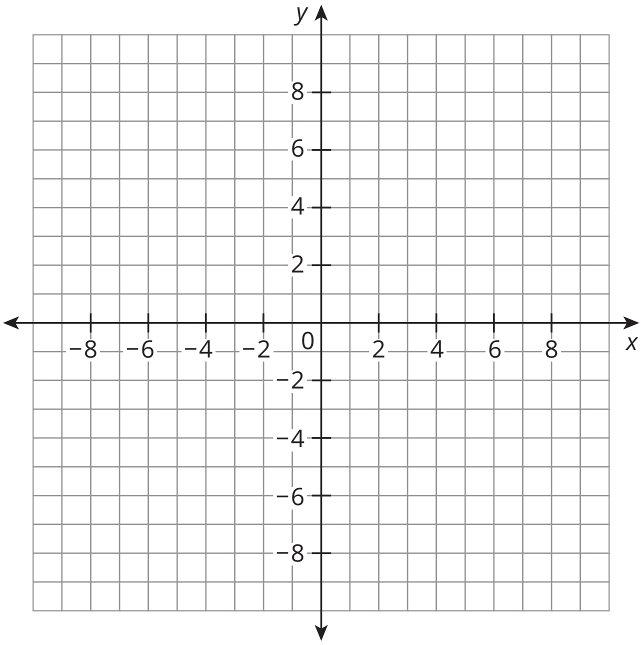
Step 2:

2. Given $y = p(x)$, sketch $m(x) = -p(x + 3)$. Describe the transformations you performed.

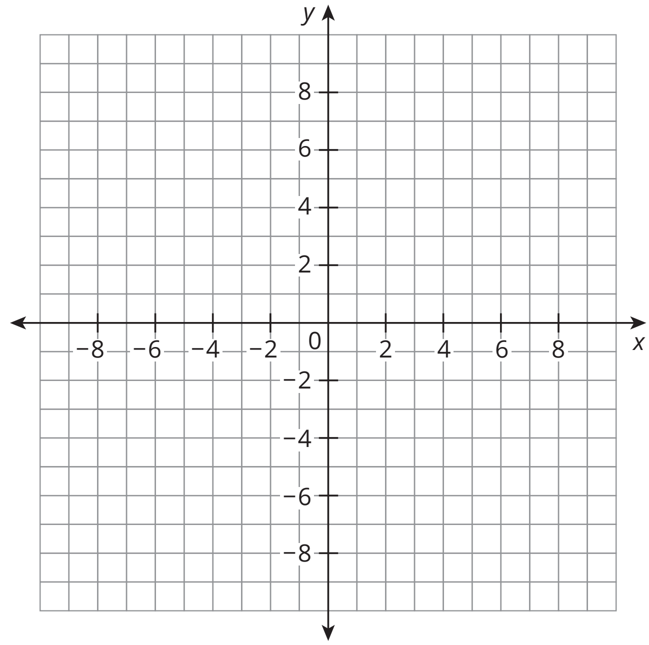


3. Given $f(x) = x^2$, graph each function. Then, write each corresponding quadratic equation.

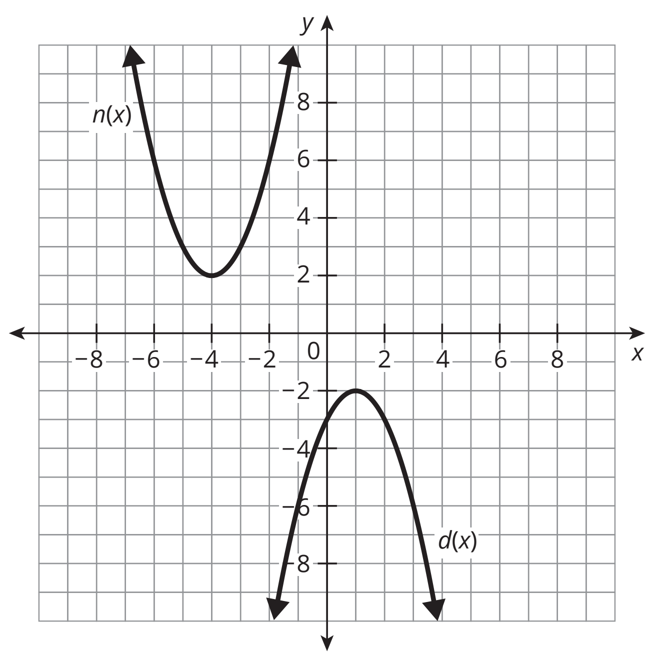
a. $f'(x) = \frac{1}{2}f(x - 2) + 3$



b. $f'(x) = -3f(x + 1) + 1$



4. Write $n(x)$ in terms of $d(x)$. Then, write the quadratic equation for $n(x)$.



ACTIVITY
4.3

Vertex Form of a Quadratic Function

Given a parent function $y = f(x)$, you have learned how to identify the effects and graph a function written in the transformation form $g(x) = af(x - c) + d$. For quadratic functions written in transformation form, $a \neq 0$.

For quadratic functions specifically, you will also see them written in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. This is referred to as **vertex form**.

In vertex form, the coefficient of x is always 1. Therefore, the B-value in the transformation form in this case is also 1 and is left out of the expression.

1. What does the variable h represent in the vertex form of a quadratic function?

Think About . . .
Do you see how this form of the function tells you about the vertex?

2. What does the variable k represent in the vertex form of a quadratic function?

3. What key characteristics can you determine directly from the quadratic function when it is written in vertex form?

4. Koda, Huyen, Angel, Destiny, and Luis are working together to write a quadratic function to represent a parabola that opens upward and has a vertex at (−6, −4).

Koda (thumbs up) My function is $s(x) = 3(x + 6)^2 - 4$.	**Huyen** (thumbs up) My function is $t(x) = \frac{1}{4}(x + 6)^2 - 4$.
Angel (thumbs down) My function is $j(x) = -3(x + 6)^2 - 4$.	**Destiny** (thumbs up) My function is $D(x) = (x + 6)^2 - 4$.
Luis (thumbs down) My function is $z(x) = 2(x - 6)^2 - 4$.	

a. What are the similarities among all the graphs of the functions? What are the differences among the graphs?

b. How is it possible to have more than one correct function?

c. What would you tell Angel and Luis to correct their functions?

d. How many possible functions can you write for the parabola described in this problem? Explain your reasoning.

5. Use technology to graph each function. Then, use the graph to rewrite the function in vertex form and in factored form.

a. $h(x) = x^2 - 8x + 12$

- Vertex: ______________________
- Vertex form: ______________________
- Zero(s): ______________________
- Factored form: ______________________

b. $r(x) = -2x^2 + 6x + 20$

- Vertex: ______________________
- Vertex form: ______________________
- Zero(s): ______________________
- Factored form: ______________________

c. $w(x) = -x^2 - 4x$

- Vertex: ______________________
- Vertex form: ______________________
- Zero(s): ______________________
- Factored form: ______________________

d. $c(x) = 3x^2 - 3$

- Vertex: ______________________
- Vertex form: ______________________
- Zero(s): ______________________
- Factored form: ______________________

6. Identify the form(s) of each quadratic function as either standard form, factored form, or vertex form. Then, state everything you know about each quadratic function's key characteristics, based only on the given equation of the function.

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

a. $g(x) = -(x - 1)^2 + 9$

b. $g(x) = x^2 + 4x$

c. $g(x) = -\frac{1}{2}(x - 3)(x + 2)$

d. $g(x) = x^2 - 5$

ACTIVITY 4.4

Writing Equations in Vertex and Factored Forms

You can write a quadratic function in vertex form when you know the coordinates of the vertex and another point on the graph.

WORKED EXAMPLE

Write an equation for a quadratic function with vertex (1, −2) that passes through the point (0, 1).

Step 1: Substitute the coordinates of the vertex into vertex form of a quadratic function.

$y = a(x - h)^2 + k$
$y = a(x - 1)^2 - 2$

Step 2: Substitute the coordinates of the other point on the graph for *x* and *y*.

$1 = a(0 - 1)^2 - 2$

Step 3: Solve for the value of *a*.

$1 = a(-1)^2 - 2$
$1 = a(1) - 2$
$1 = a - 2$
$3 = a$

Step 4: Rewrite the equation in vertex form, substituting the vertex and the value of *a*.

$f(x) = 3(x - 1)^2 - 2$

1. How would you determine an equation of a quadratic function in factored form given the zeros and another point on the graph?

2. Malik and Michael each wrote an equation for the function represented by the graph shown.

Malik

$y = a(x + 1)^2 - 3$

$0 = a(1 + 1)^2 - 3$

$0 = 4a - 3$

$3 = 4a$

$a = \frac{3}{4}$

$y = \frac{3}{4}(x + 1)^2 - 3$

Michael

$y = a(x + 3)(x - 1)$

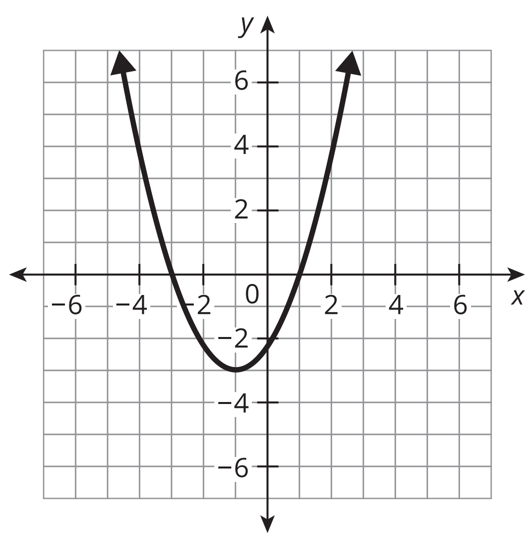
$-3 = a(-1 + 3)(-1 - 1)$

$-3 = a(2)(-2)$

$-3 = -4a$

$a = \frac{3}{4}$

$y = \frac{3}{4}(x + 3)(x - 1)$



a. Explain Malik's reasoning.

b. Explain Michael's reasoning.

c. Use technology to show that Mailk's equation and Michael's equation are equivalent.

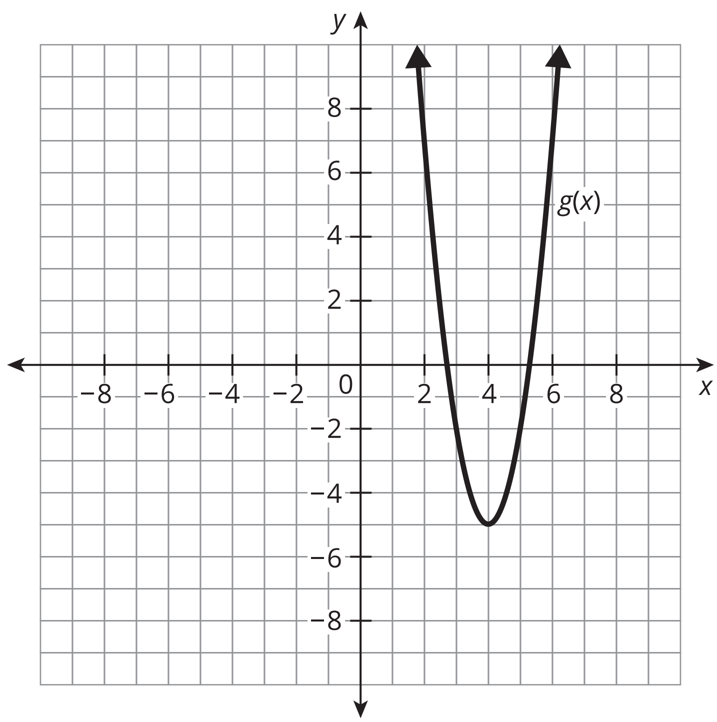
Ask Yourself . . .
How does representing mathematics in multiple ways help to communicate reasoning?

3. Write an equation for a quadratic function in vertex form with vertex (3, 1) that passes through the point (1, 9).

4. Write an equation for a quadratic function in factored form with zeros at $x = -4$ and $x = 0$ that passes through the point $(-3, 6)$.

5. Write an equation for a quadratic function in vertex form with vertex $(-1, 6)$ that passes through the point $(-3, 4)$.

6. Write an equation for a quadratic function $g(x)$ in vertex form given the graph of $g(x)$.



Talk the Talk

Show What You Know

1. Based on the equation of each function, describe how the graph of each function compares to the graph of $f(x) = x^2$.

   a. $z(x) = -(x - 1)^2 - 10$

   b. $r(x) = \frac{1}{2}(x + 6)^2 + 7$

   c. $m(x) = (4x)^2 + 5$

2. Describe each transformation in relation to the parent function $f(x) = x^2$.

   a. $h(x) = f(x) + d$, when $d > 0$

   b. $h(x) = f(x) + d$, when $d < 0$

   c. $h(x) = f(x - c)$, when $c > 0$

d. $h(x) = f(x - c)$, when $c < 0$

e. $h(x) = af(x)$, when $|a| > 1$

f. $h(x) = af(x)$, when $0 < |a| < 1$

g. $h(x) = af(x)$, when $a = -1$

Lesson 4 Assignment

Write

Describe the connections between the vertex form, $f(x) = a(x - h)^2 + k$, and the transformation form, $g(x) = a \cdot f(x - c) + d$, of the parent quadratic function, $y = f(x)$.

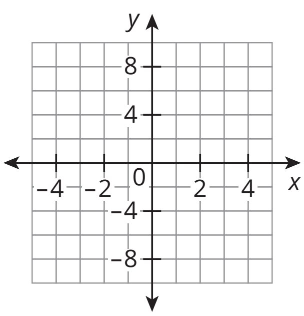
Remember

Transformations performed on any function $f(x)$ can be described by the transformation function $g(x) = af(b(x - c)) + d$ where the c-value translates the function $f(x)$ horizontally, the d-value translates $f(x)$ vertically, the a-value vertically stretches or compresses $f(x)$, and the b-value horizontally stretches or compresses $f(x)$. When the a-value is negative, the function $f(x)$ is reflected across a horizontal line of reflection, and when the b-value is negative, the function $f(x)$ is reflected across a vertical line of reflection.

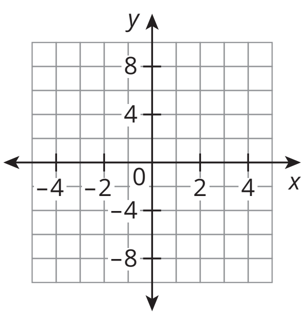
Practice

1. Given $f(x) = x^2$, graph each function and write the corresponding quadratic equation.

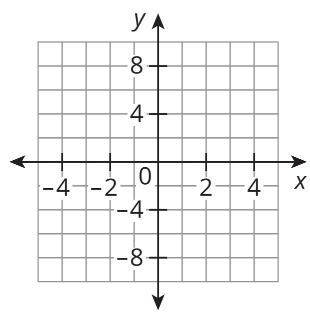
   a. $g(x) = 3f(x - 1)$



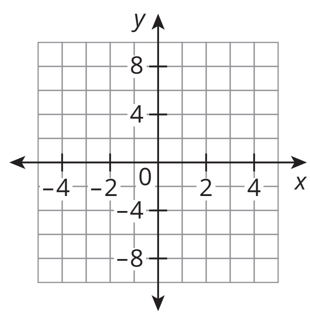
   b. $g(x) = f(3x) - 1$



   c. $g(x) = \frac{1}{2} f(x) + 5$



   d. $g(x) = 2f(x - 3) + 1$



Lesson 4 Assignment

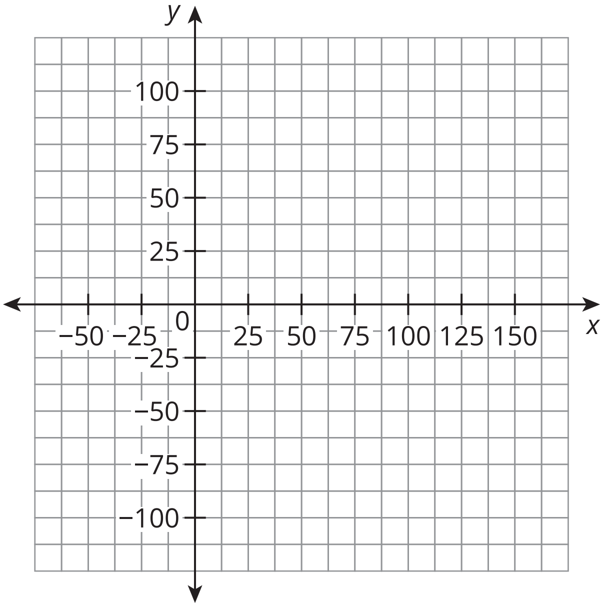
2. The graph shows the parent function $f(x) = x^2$ as well as the function $h(x)$.

   c. Describe the types of transformations performed on $f(x)$ to result in $h(x)$.

   b. When the dilation factor is 16, write the function $h(x)$.

3. Use the given characteristics to write a function $R(x)$ in vertex form. Then, sketch the graph of $R(x)$ and the parent function $f(x) = x^2$.

   - The function has an absolute maximum.
   - The function is translated 70 units up and 100 units to the right.
   - The function is vertically dilated by a factor of $\frac{1}{5}$.



Prepare

Rewrite each expression by combining like terms.

1. $-3x + 4y - 9x - 5y$

2. $2xy^2 + 5x^2y - 7xy + xy^2$

3. $6 - m^2 + 5m^2$

4. $-8 - (-4k) + 7 + 1 - 4k$

TOPIC 1 SELF-REFLECTION

Name: ______________________________

Introduction to Quadratic Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Introduction to Quadratic Functions* topic by:

TOPIC 1: *Introduction to Quadratic Functions*	Beginning of Topic	Middle of Topic	End of Topic
defining the terms and coefficients of a quadratic equation.	○	○	○
using context to define the variables and quantities for a quadratic function.	○	○	○
recognizing real-world problem situations that can be best represented with quadratic functions.	○	○	○
identifying the defining characteristics of the graph of a quadratic function and the equation it represents.	○	○	○
interpreting key characteristics of a graph of a quadratic function in terms of a given problem situation.	○	○	○
graphing quadratic functions on the coordinate plane and using the graph to identify key attributes, when possible, including the *x*-intercept, *y*-intercept, zeros, maximum or minimum value, vertex, and the equation of the axis of symmetry.	○	○	○
determining the domain and range of quadratic functions.	○	○	○

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Introduction to Quadratic Functions*	Beginning of Topic	Middle of Topic	End of Topic
recognizing the different forms of a quadratic function, i.e., standard form, factored form, and vertex form.	○	○	○
rewriting a quadratic function from one form to another using graphs and technology.	○	○	○
writing the equation of a quadratic function given the vertex and another point on the graph.	○	○	○
evaluating a quadratic function for a given value.	○	○	○
using second differences to determine whether a table of values represents a quadratic function.	○	○	○
understanding that vertex form is a special name given to transformations formed on the parent function $f(x) = x^2$.	○	○	○
transforming a quadratic function of the form $y = a \cdot f(b(x - c)) + d$ and understanding how the a-, b-, c-, and d-values affect the original function.	○	○	○
graphing the transformation of a given quadratic function using reference points.	○	○	○

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Introduction to Quadratic Functions* topic.

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2. What mathematical understandings from the topic do you feel you are making the most progress with?

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3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

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TOPIC 1 SUMMARY

Introduction to Quadratic Functions Summary

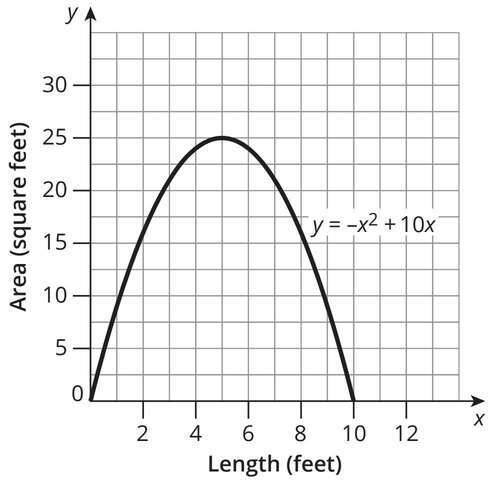
LESSON 1 Exploring Quadratic Functions

The shape that a quadratic function forms when graphed is called a *parabola*. A **parabola** is a smooth U-shaped curve that has symmetry. The parabola can open upward, decreasing to a minimum point before increasing, or can open downward, increasing to a maximum point before decreasing. The domain of a quadratic function is all real numbers. The range of a quadratic function is all real numbers greater than or equal to the minimum *y*-value or less than or equal to the maximum *y*-value.

The graph of a quadratic function has one *y*-intercept and, at most, two *x*-intercepts.

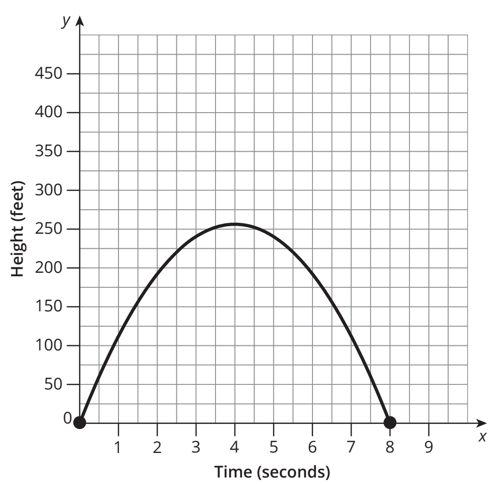
Quadratic functions model area because area is measured in square units.

For example, suppose you have 20 feet of fencing with which to enclose a rectangular area.



The graph represents a quadratic function for the area of the rectangle given possible lengths of the rectangle. The maximum of the parabola is at the point (5, 25). It has *x*-intercepts at (0, 0) and (10, 0) and a *y*-intercept at (0, 0).

A **vertical motion model** is a quadratic equation that models the height of an object at a given time. The equation is of the form $y = -16t^2 + v_0t + h_0$, where *y* represents the height of the object in feet, *t* represents the time in seconds that the object has been



NEW KEY TERMS

- parabola [parábola]
- vertical motion model [modelo de movimiento vertical]
- roots [raíces]
- second differences [segundas diferencias]
- standard form of a quadratic function [forma estándar de una función cuadrática]
- factored form [forma factorizada]
- concave down
- concave up
- vertex [vértice]
- axis of symmetry [eje de simetría]
- argument of a function [argumento de una función]
- reflection [reflexión]
- line of reflection [línea de reflexión]
- vertex form [forma de vértice]

moving, v_0 represents the initial vertical velocity of the object in feet per second, and h_0 represents the initial height of the object in feet.

For example, suppose a firework is launched into the air from the ground with a vertical velocity of 128 feet per second. The function that describes the height of the firework in terms of time is $g(t) = -16t^2 + 128t$.

The x-intercepts of a graph of a quadratic function are also called the *zeros* of the quadratic function. The x-intercepts, or zeros, of $g(t) = -16t^2 + 128t$ are (0, 0) and (8, 0).

When an equation is used to model a situation, the x-intercepts are referred to as roots. The **roots** of an equation indicate where the graph of the equation crosses the x-axis.

LESSON 2 Key Characteristics of Quadratic Functions

First differences are the differences between successive output values when successive input values have a difference of 1. **Second differences** are the differences between consecutive values of first differences. Linear functions have constant first differences and second differences of 0. Quadratic functions have changing first differences and constant second differences.

Linear function: $f(x) = -x + 2$

x	$f(x)$
0	2
1	1
2	0
3	−1
4	−2

First Differences	Second Differences
−1	0
−1	0
−1	0
−1	

Quadratic function: $f(x) = 2x^2 - 3x$

x	$f(x)$
0	0
1	−1
2	2
3	9
4	20

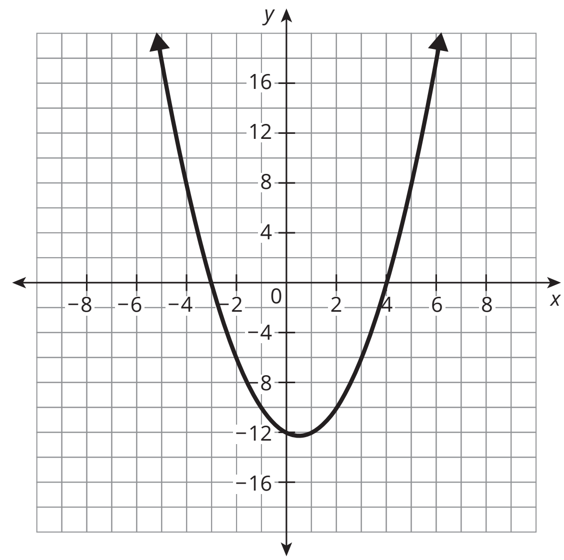
First Differences	Second Differences
−1	4
3	4
7	4
11	

The **standard form of a quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$. In this form, a and b represent numerical coefficients and c represents a constant. A quadratic function written in **factored form** is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$ and r_1 and r_2 represent the roots.

When the leading coefficient a is negative, the graph of the quadratic function is identified as **concave down**, which means it opens downward and has a maximum. When a is positive, the graph of the quadratic function is identified as **concave up**, which means it opens upward and has a minimum. When a quadratic function is written in standard form, the constant c is the y-intercept.

The **vertex** of a parabola is the lowest or highest point on the graph of the quadratic function. The **axis of symmetry** of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images.

For a quadratic function in factored form, the equation for the axis of symmetry is given by $x = \frac{r_1 + r_2}{2}$. For a quadratic function in standard form, the equation for the axis of symmetry is $x = \frac{-b}{2a}$.



The graph represents the function $f(x) = x^2 - x - 12$. The axis of symmetry is $x = -\frac{(-1)}{2(1)} = \frac{1}{2}$. The vertex is $\left(\frac{1}{2}, -12\frac{1}{4}\right)$. The x-intercepts, or zeros, of the function are $x = -3$ and $x = 4$, so the function can be written in factored form as $f(x) = (x + 3)(x - 4)$. The y-intercept is $(0, -12)$. The domain of the function is all real numbers, and the range is all real numbers greater than or equal to $-12\frac{1}{4}$. The graph has an interval of decrease from $-\infty$ to $\frac{1}{2}$ and an interval of increase from $\frac{1}{2}$ to ∞.

You can use the fact that the graph of a quadratic function is symmetric about the axis of symmetry to determine a second point on the parabola given a point on the parabola and to determine the axis of symmetry given two symmetric points on the parabola.

For example, the vertex of a parabola is (3, 5). A point on the parabola is (0, 3). Another point on the parabola is (6, 3).

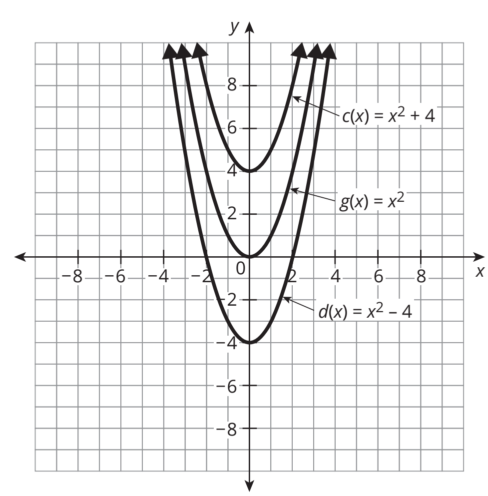
$$\frac{0 + a}{2} = 3$$
$$0 + a = 6$$
$$a = 6$$

Suppose two symmetric points on a parabola are (−7, 20) and (4, 20). The axis of symmetry is $x = -\frac{3}{2}$ because $\frac{-7 + 4}{2} = -\frac{3}{2}$.

LESSON 3 Quadratic Function Transformations

You can use function transformation form, $a \cdot f(b(x - c)) + d$, to transform quadratic functions.

The function $f(x)$ of the form $f(x) = g(x) + d$ is a vertical translation of the function $g(x)$. The value $|d|$ describes how many units up or down the graph of the original function is translated. Vertical translations are performed on a parent quadratic function $g(x) = x^2$ by adding a constant to, or subtracting a constant from, the function. Adding to the function translates it up, and subtracting translates it down.

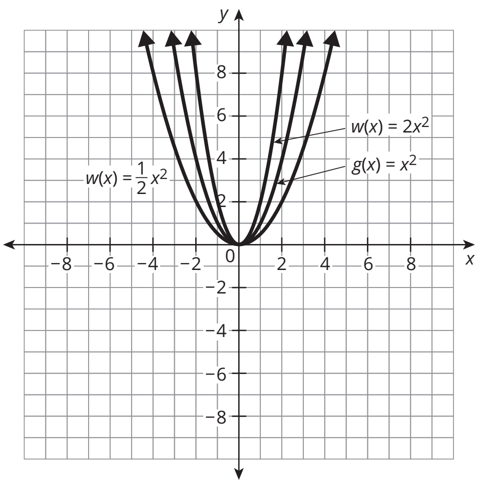


Vertical translations

$g(x) = x^2$	Parent function
$c(x) = g(x) + 4$	$g(x)$ translated 4 units up, so $(x, y) \longrightarrow (x, y + 4)$
$d(x) = g(x) - 4$	$g(x)$ translated 4 units down, so $(x, y) \longrightarrow (x, y - 4)$

The function $f(x)$ in the form $f(x) = a \cdot g(x)$ is a vertical dilation of the function $g(x)$. The a-value describes the vertical dilation of the graph of the original function. A **vertical dilation** of a function is a transformation in which the y-coordinate of every point on the graph of the function is multiplied by a common factor called the **dilation factor**. A vertical dilation stretches or shrinks the graph of a function vertically. For the transformed parent quadratic function, $f(x) = a \cdot g(x)$, when $|a| > 1$, the graph of $g(x)$ is stretched vertically. When $0 < |a| < 1$, $g(x)$ shrinks vertically. You can use the coordinate notation shown to indicate a vertical dilation.

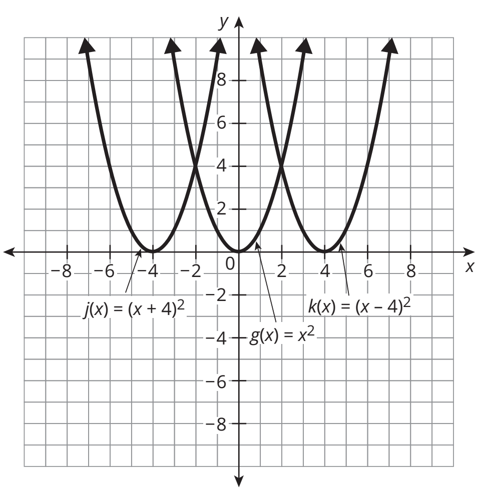
$(x, y) \longrightarrow (x, ay)$, where a is the dilation factor.



Vertical dilations

$g(x) = x^2$	Parent function
$v(x) = 2g(x)$	$g(x)$ stretched by a dilation factor of 2, so $(x, y) \longrightarrow (x, 2y)$
$w(x) = \frac{1}{2}g(x)$	$g(x)$ shrunk by a dilation factor of $\frac{1}{2}$, so $(x, y) \longrightarrow (x, \frac{1}{2}y)$

Horizontal translations are performed on the parent quadratic function $g(x) = x^2$ by adding a constant to or subtracting a constant from the argument, x, of the function. The **argument of a function** is the expression inside the parentheses. For $f(x) = g(x - c)$, the expression $x - c$ is the argument of the function. The c-value of the transformed function affects the input values of the function. The value $|c|$ describes the number of units the graph of $g(x)$ is translated right or left. When $c > 0$, the graph is translated to the right. When $c < 0$, the graph is translated to the left.

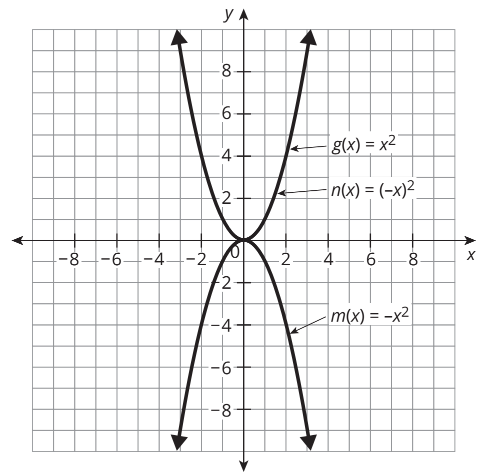


Horizontal translations

$g(x) = x^2$	Parent function
$j(x) = g(x + 4)$	$g(x)$ translated 4 units left, so $(x, y) \longrightarrow (x - 4, y)$
$k(x) = g(x - 4)$	$g(x)$ translated 4 units right, so $(x, y) \longrightarrow (x + 4, y)$

Changing the *a*-value of a function to its opposite reflects the function across a horizontal line. A **reflection** of a graph is the mirror image of the graph about a line of reflection. The line that the graph is reflected across is called the **line of reflection**. A horizontal line of reflection affects the *y*-coordinates, and a vertical line of reflection affects the *x*-coordinates. The line of reflection for the function might be different, depending on how you write the transformation and the order in which the transformations are applied.

Multiplying the parent quadratic function, $g(x) = x^2$, by -1 results in a reflection across the line $y = 0$. Multiplying the argument of the parent quadratic function by -1 results in a reflection across the line $x = 0$, which ends up being the same as the original function because quadratic functions have a vertical axis of symmetry.



Reflections

$g(x) = x^2$ Parent function

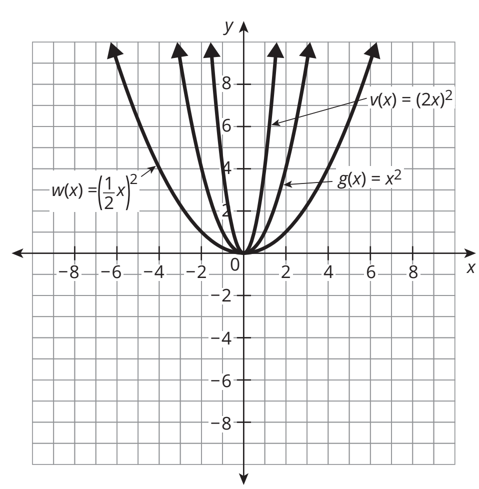
$m(x) = -g(x)$ $g(x)$ is reflected across $y = 0$, so $(x, y) \longrightarrow (x, -y)$

$n(x) = g(-x)$ $g(x)$ is reflected across $x = 0$, so $(x, y) \longrightarrow (-x, y)$

LESSON 4 Horizontal Transformations and Vertex Form

A **horizontal dilation** is a type of transformation that stretches or compresses the entire graph. **Horizontal stretching** is the stretching of a graph away from the *y*-axis. **Horizontal compression** is the squeezing of a graph towards the *y*-axis. A horizontal dilation of a function is a transformation in which the *x*-coordinate of every point on the graph of the function is multiplied by a dilation factor. For the transformed parent quadratic function, $f(x) = g(bx)$, when $|b| > 1$, the graph of the function is compressed horizontally. When $0 < |b| < 1$, the function is stretched horizontally. You can use the coordinate notation shown below to indicate a horizontal dilation.

$(x, y) \longrightarrow \left(\frac{1}{|b|}x, y\right)$, where $\frac{1}{|b|}$ is the dilation factor.

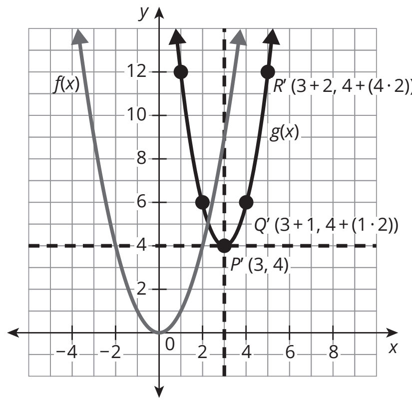


Horizontal dilations

$g(x) = x^2$ Parent function

$v(x) = g(2x)$ $g(x)$ compressed by a dilation factor of $\frac{1}{2}$, so $(x, y) \longrightarrow \left(\frac{1}{2}x, y\right)$

$w(x) = g\left(\frac{1}{2}x\right)$ $g(x)$ stretched by a dilation factor of 2, so $(x, y) \longrightarrow (2x, y)$



Given $y = f(x)$ is the parent function, you can use reference points to graph $y = a \cdot f(b(x - c)) + d$ without using technology. Any point (x, y) on $f(x)$ maps to the point $\left(\frac{1}{b}x + c, ay + d\right)$.

Given $f(x) = x^2$, the function $g(x) = 2f(x - 3) + 4$ has been graphed.

A quadratic function written in **vertex form** is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The variable h represents the x-coordinate of the vertex. The variable k represents the y-coordinate of the vertex.

You can write a quadratic function in vertex form when you know the coordinates of the vertex and another point on the graph.

For example, write an equation for a quadratic function with the vertex $(1, -2)$ that passes through the point $(0, 1)$.

Step 1: Substitute the coordinates of the vertex into vertex form of a quadratic function.

$$y = a(x - h)^2 + k$$
$$y = a(x - 1)^2 - 2$$

Step 2: Substitute the coordinates of the other point on the graph for x and y.

$$1 = a(0 - 1)^2 - 2$$

Step 3: Solve for the value of a.

$$1 = a(-1)^2 - 2$$
$$1 = a(1) - 2$$
$$1 = a - 2$$
$$3 = a$$

Step 4: Rewrite the equation in vertex form, substituting the vertex and the value of a.

$$f(x) = 3(x - 1)^2 - 2$$

TOPIC 2

A parabola is the shape a quadratic function forms when graphed. It is a smooth curve with reflectional symmetry. The arches in the bridge over Lady Bird Lake in Austin, Texas, are parabolas.

Polynomial Operations

LESSON 1 Adding and Subtracting Polynomials **759**

LESSON 2 Multiplying Polynomials **781**

LESSON 3 Polynomial Division **801**

1 Adding and Subtracting Polynomials

OBJECTIVES

- Name polynomials by number of terms or degree.
- Understand that you can perform operations on functions as well as numbers.
- Add and subtract polynomials.

NEW KEY TERMS

- polynomial
- monomial
- binomial
- trinomial
- degree of a polynomial

You know that a linear expression is one type of polynomial expression.

What are other polynomial expressions, and how do you add and subtract them?

Getting Started

Sorting It Out

You are familiar with many types of mathematical expressions. Cut out the 12 expressions located at the end of this lesson. Analyze and sort them into groups based on common characteristics.

1. Summarize the groups you formed by listing the expressions that you grouped together and your description for each group. Use mathematical terms in your descriptions.

2. Compare your groups of expressions to your classmates' groups. Describe any similarities and differences.

3. Michael and Javier agree that $4x - 6x^2$ and $25 - 18m^2$ belong in the same group. They each are adding the expressions shown to the group. Who is correct? Explain your reasoning.

Michael	Javier
$5 - 7h$	$y^2 - 4y + 10$
$78j^3 - 3j$	$-3 + 7n + n^2$
$-13s + 6$	

4. What characteristics do all twelve expressions share?

ACTIVITY 1.1

Categorizing Polynomials

Previously, you worked with linear expressions in the form $ax + b$ and quadratic expressions in the form $ax^2 + bx + c$. Each is also part of a larger group of expressions known as *polynomials*.

A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k, where a is any real number and k is a non-negative integer. In general, a polynomial is of the form $a_1x^k + a_2x^{k-1} + \ldots + a_nx^0$. Within a polynomial, each product is a term and the number being multiplied by a power is a coefficient.

WORKED EXAMPLE

The polynomial $m^3 + 8m^2 - 10m + 5$ has four terms. Each term is written in the form ax^k.

- The first term is m^3.
- The power is m^3, and its coefficient is 1.
- In this term, the variable is m and the exponent is 3.

1. Write each term from the Worked Example and identify the coefficient, the power, and the exponent. The first term has already been completed for you.

	1st	2nd	3rd	4th
Term	m^3			
Coefficient	1			
Variable	m			
Power	m^3			
Exponent	3			

2. Identify the terms and coefficients in each polynomial.

   a. $-2x^2 + 100x$

   b. $4m^3 - 2m^2 - 5$

   c. $y^5 - y + 3$

Polynomials are named according to the number of terms they have. Polynomials with only one term are **monomials**. Polynomials with exactly two terms are **binomials**. Polynomials with exactly three terms are **trinomials**.

The degree of a term in a polynomial is the exponent of the term. The greatest exponent in a polynomial determines the **degree of the polynomial**. In the polynomial $4x + 3$, the greatest exponent is 1, so the degree of the polynomial is 1.

3. Fernando says that $3x^{-2} + 4x - 1$ is a trinomial with a degree of 1 because 1 is the greatest exponent. Olivia disagrees and says that this is not a polynomial because the power on the first term is not a whole number. Who is correct? Explain your reasoning.

4. Determine whether each expression is a polynomial. Explain your reasoning.

   $5^x + 4^{x-1} + 3^{x-2}$ $\qquad$ $x^2 + \sqrt{x}$ $\qquad$ $x^4y + x^3y^2 + x^2y$

A polynomial is written in standard form when the terms are in descending order, starting with the term with the largest degree and ending with the term with the smallest degree.

5. Revisit the cards you sorted in the Getting Started.

   a. Identify any polynomial not written in standard form and rewrite it in standard form on the card.

   b. Identify the degree of each polynomial and write the degree on the card.

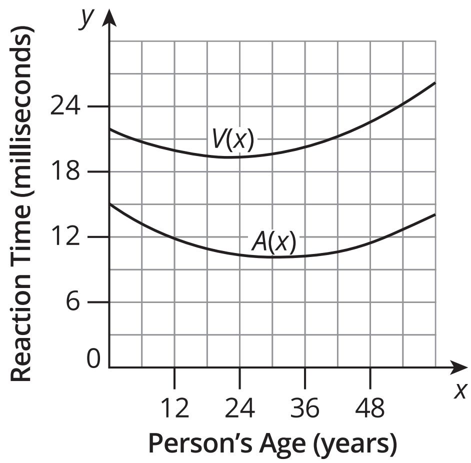
   c. Glue each card in the appropriate column based on the number of terms in each polynomial. Write your own polynomial to complete any empty boxes.

Monomial	Binomial	Trinomial

ACTIVITY 1.2

Interpreting the Graphs of Polynomial Functions

The graphs of functions $V(x)$ and $A(x)$ are shown. The function $V(x)$ models people's reaction times to visual stimuli in milliseconds based on the age of a person in years. The function $A(x)$ models people's reaction times to audio stimuli in milliseconds based on the age of a person in years.



1. Interpret the graphs of the functions.

   a. Describe the functions $V(x)$ and $A(x)$.

   b. Write a summary to describe people's reaction times to visual stimuli and audio stimuli.

   c. Do you think a person would react faster to a car horn or a flashing light? Explain your reasoning.

Ask Yourself . . .
How can you incorporate information about auto insurance rates and a driver's age in your report?

2. Estimate the age that a person has the quickest reaction time to each stimuli. Explain how you determined each answer.

   a. Visual stimuli

   b. Audio stimuli

Many times, auto insurance companies use test results similar to the ones shown to create insurance policies for different drivers.

3. How do you think the information provided in the graphic representation may be used by an auto insurance company?

4. Consider a new function $h(x)$, where $h(x) = V(x) - A(x)$. What does $h(x)$ mean in terms of the problem situation?

5. Write a report about drivers' reaction times to visual and audio stimuli. Discuss actions that may improve drivers' reaction times and distractions that may worsen drivers' reaction times. Discuss the importance of flashing lights and sirens on emergency vehicles.

ACTIVITY 1.3

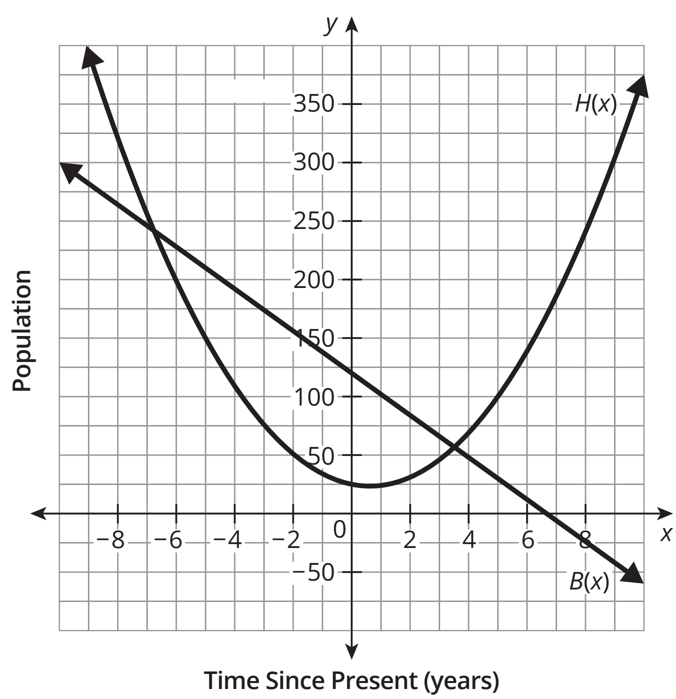
Adding and Subtracting Polynomial Functions

You are playing a new virtual reality game called *Species*. You are an environmental scientist who is responsible for tracking two species of endangered parrots, the orange-bellied parrot and the yellow-headed parrot. Suppose the orange-bellied parrots' population can be modeled by the function $B(x)$, where x represents the number of years since the current year. Suppose that the population of the yellow-headed parrot can be modeled by the function $H(x)$.

$$B(x) = -18x + 120$$

$$H(x) = 4x^2 - 5x + 25$$

The two polynomial functions are shown on the coordinate plane.



Your new task in this game is to determine the total number of these endangered parrots each year over a six-year span. You can calculate the total population of parrots using the two graphed functions.

1. Use the graphs of $B(x)$ and $H(x)$ to determine the function, $T(x)$, to represent the total population of parrots.

   a. Write $T(x)$ in terms of $B(x)$ and $H(x)$.

Ask Yourself . . .
One place to start the sketch of $T(x)$ would be to consider the y-intercept for each function. What would the new y-intercept be for $T(x)$?

b. Predict the shape of the graph of $T(x)$.

c. Sketch a graph of $T(x)$ on the coordinate plane shown. First, choose any 5 x-values and add their corresponding y-values to create a new point on the graph of $T(x)$. Then, connect the points with a smooth curve. Record the values in the table.

x	$B(x)$	$H(x)$	$T(x)$

d. Did the graph of $T(x)$ match your prediction in part (b)? Identify the function family to which $T(x)$ belongs.

You can write a function, $T(x)$, in terms of x to calculate the total number of parrots at any time.

WORKED EXAMPLE

$T(x) = B(x) + H(x)$	Write $T(x)$ in terms of two known functions.
$T(x) = (-18x + 120) + (4x^2 - 5x + 25)$	Substitute the functions in terms of x.
$T(x) = 4x^2 + (-18x + (-5x)) + (120 + 25)$	Use the commutative property to reorder and the associative property to group like terms.
$T(x) = 4x^2 - 23x + 145$	Combine like terms.

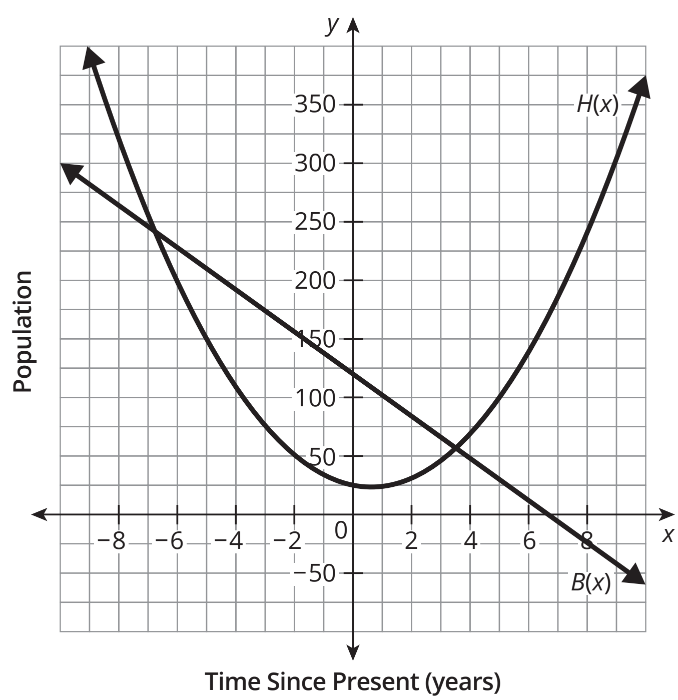
Remember . . .
The table feature on a graphing calculator is an efficient tool to determine y-values.

2. Choose any two x-values in your table. Use the new polynomial function, $T(x)$, to confirm that your solution in the table for those times is correct. Show your work.

3. Use technology to confirm that your graph and the remaining solutions in the table are correct. Explain any discrepancies and how you corrected them.

4. Daniela says that using $T(x)$ will not work for any time after 6 years from now because by that point the orange-bellied parrot will be extinct. Is Daniela's statement correct? Why or why not?

Throughout the game *Species*, you must always keep track of the difference between the population of each type of species. If the difference gets to be too great, you lose the game. The graphs of $B(x) = -18x + 120$ and $H(x) = 4x^2 - 5x + 25$ are shown.



5. Use the graphs of $B(x)$ and $H(x)$ to determine the function $D(x)$ to represent the difference between the populations of each type of species.

   a. Write $D(x)$ in terms of $B(x)$ and $H(x)$.

   b. Predict the shape of the graph of $D(x)$.

c. Sketch a graph of $D(x)$ on the coordinate plane shown. First, choose any 5 x-values and subtract their corresponding y-values to create a new point on the graph of $D(x)$. Then, connect the points with a smooth curve. Record the values in the table.

x	$B(x)$	$H(x)$	$D(x)$

d. Did the graph of $D(x)$ match your prediction in part (b)? Identify the function family to which $D(x)$ belongs.

Think About . . .

Refer to the Worked Example for adding polynomials as a guide.

6. Write a function, $D(x)$, in terms of x to calculate the difference between the population of the orange-bellied parrots and the yellow-headed parrots. Write $D(x)$ as a polynomial in standard form.

7. Choose any two x-values in your table. Use your new polynomial function to confirm that your solution in the table for those times is correct. Show your work.

8. Use technology to confirm that your graph and the remaining solutions in the table are correct. Explain any discrepancies and how you corrected them.

9. Alexander uses his function $D(x) = -4x^2 - 13x + 95$ to determine that the difference between the number of orange-bellied parrots and the number of yellow-headed parrots 7 years from now will be -192. Is Alexander correct or incorrect? If he is correct, explain to him what his answer means in terms of the problem situation. If he is incorrect, explain where he made his error and how to correct it.

10. The next round of the *Species* game included the red-winged parrot, whose population can be modeled by the function $W(x) = -9x + 80$, and the rainbow lorikeet parrot, whose population can be modeled by the function $L(x) = 2x^2 - 4x + 10$. In both cases, x represents the number of years since the current year.

   a. Write a function, $S(x)$, in terms of x to calculate the total number of red-winged parrots and rainbow lorikeet parrots at any time.

   b. Write a function, $M(x)$, in terms of x to calculate the difference in the number of red-winged parrots and rainbow lorikeet parrots at any time.

   c. Calculate $S(4)$ and $M(4)$. Interpret the meaning of your results.

   d. In four years, how many red-winged parrots will there be? How many rainbow lorikeet parrots will there be?

ACTIVITY

1.4 Just the Math

In this activity, you will practice adding and subtracting polynomials.

WORKED EXAMPLE

You can add and subtract polynomials by grouping terms that have the same variable and exponent.

To determine the sum $(3x^2 - 2x + 1) + (2x^2 + 5x - 5)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols. $3x^2 - 2x + 1 + 2x^2 + 5x - 5$

Step 2: Use the commutative property of Addition to rearrange the terms. $3x^2 + 2x^2 - 2x + 5x + 1 - 5$

Step 3: Use the associative property of addition to group like terms. $(3x^2 + 2x^2) + (-2x + 5x) + (1 - 5)$

Step 4: Simplify the like terms. $5x^2 + 3x - 4$

To determine the difference $(2x^2 - 4x + 3) - (5x^2 - 3x + 6)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols. $2x^2 - 4x + 3 - 5x^2 + 3x - 6$

Step 2: Rewrite the terms using addition. $2x^2 + (-4x) + 3 + (-5x^2) + 3x + (-6)$

Step 3: Use the commutative property of addition to rearrange the terms. $2x^2 + (-5x^2) + (-4x) + 3x + 3 + (-6)$

Step 4: Use the associative property of addition to group like terms. $(2x^2 + (-5x^2)) + ((-4x) + 3x) + (3 + (-6))$

Step 5: Simplify the grouped terms. $-3x^2 - x - 3$

1. Analyze each student's work. Determine the error and make the necessary corrections.

Jacob

$3x^2 + 5x^2 = 8x^4$

Amir

$2x - (4x + 5)$

$2x - 4x + 5$

$-2x + 5$

Destiny

$(4x^2 + 2x - 5) + (3x^2 + 7)$

$(4x^2 + 3x^2) - (2x) - (5 + 7)$

$7x^2 - 2x - 12$

2. Determine the sum or difference of each Show your work.

a. $(6ab + 3ac - 4bc) + (-ac + 3bc)$

b. $(7xy - 9yz + 2xz) - (10xy + 4yz - 3xy)$

c. $(5x^2 - 2x + 7) - (4x + 3)$

d. $(-2x^2 + 3x - 1) - (3x - 1)$

e. $(x^2 - 2x - 3) + (2x^2 + 1)$

f. $(2x^2 + 3x - 4) - (2x^2 + 5x - 6)$

g. $(4x^3 + 5x - 2) + (7x^2 - 8x + 9)$

h. $(9x^4 - 5) - (8x^4 - 2x^3 + x)$

Talk the Talk

Putting It Into Practice

Match each expression with the equivalent polynomial.

Expressions	Polynomials
1. $(x^2 - 3) + (x^2 + 2)$	A. -1
2. $(x^2 - 3) - (x^2 + 2)$	B. $-2x^2 - 1$
3. $(x^2 - 3) - (x^2 - 2)$	C. $-2x^2 - 5$
4. $(x^2 - 3) + (x^2 - 2)$	D. $2x^2 - 1$
5. $-(x^2 + 3) - (x^2 - 2)$	E. $2x^2 - 5$
6. $-(x^2 + 3) - (x^2 + 2)$	F. -5

Expression Cards

$4x - 6x^2$	$125p$	$\frac{4}{5}r^3 + \frac{2}{5}r - 1$
$-\frac{2}{3}$	$y^2 - 4y + 10$	$5 - 7h$
$-3 + 7n + n^2$	-6	$-13s + 6$
$12.5t^3$	$78j^3 - 3j$	$25 - 18m^2$

Why is this page blank?

So you can cut out the Expression Cards on the other side

Lesson 1 Assignment

Write

Match each definition with its corresponding term.

1. polynomial	a. a polynomial with only 1 term
2. term	b. the degree of the term with the greatest exponent
3. coefficient	c. a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients
4. monomial	d. a polynomial with exactly 3 terms
5. binomial	e. any number being multiplied by a variable to a power within a polynomial expression
6. trinomial	f. each product in a polynomial expression
7. degree of a term	g. a polynomial with exactly 2 terms
8. degree of a polynomial	h. the exponent of a term in a polynomial

Remember

A *polynomial* is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. You can operate with polynomial expressions just as you would with numeric expressions.

Practice

1. Koda and William each build a rocket launcher. They launch a model rocket using Koda's launcher, and on its way back down, it lands on the roof of a building that is 320 feet tall. The height of the rocket can be represented by the equation $H_1(x) = -16x^2 + 200x$, where x represents the time in seconds and $H_1(x)$ represents the height. Koda and William take the stairs to the roof of the building and relaunch the rocket using William's rocket launcher. The rocket lands back on the ground. The height of the rocket after this launch can be represented by the equation $H_2(x) = -16x^2 + 192x + 320$.

   a. Compare and contrast the polynomial functions.

Lesson 1 Assignment

b. Use technology to sketch a graph of the functions.

c. Does it make sense in terms of the problem situation to graph the functions outside of Quadrant I? Explain your reasoning.

d. Explain why the graphs of these functions do not intersect.

e. Koda believes that she can add the two functions to determine the total height of the rocket at any given time. Write a function $S(x)$ that represents the sum of $H_1(x)$ and $H_2(x)$. Show your work.

f. Is Koda correct? Explain your reasoning.

Lesson 1 Assignment

g. Subtract $H_1(x)$ from $H_2(x)$ and write a new function, $D(x)$, that represents the difference. Then, explain what this function means in terms of the problem situation.

2. Determine whether each expression is a polynomial. If so, identify the terms, coefficients, and degree of the polynomial. If not, explain your reasoning.

a. $-2b^4 + 4b - 1$

b. $6 - g^{-2}$

c. $8h^4$

d. $9w - w^3 + 5w^2$

e. $x^{\frac{1}{2}} + 2$

f. $\frac{4}{5}y + \frac{2}{3}y^2$

Lesson 1 Assignment

3. Given $A(x) = x^2 - 5x + 4$, $B(x) = 2x^2 + 5x - 6$, and $C(x) = -x^2 + 3$, determine each function. Write your answer in standard form.

   a. $D(x) = B(x) + C(x)$

   b. $E(x) = A(x) + B(x)$

   c. $F(x) = A(x) - C(x)$

   d. $G(x) = C(x) - B(x)$

   e. $H(x) = A(x) + B(x) - C(x)$

   f. $J(x) = B(x) - A(x) + C(x)$

4. Determine each sum or difference.

   a. $(8x - 4) + (9x + 12)$

   b. $(10p^2 - 5p + 2) - (6p^2 - 4p)$

   c. $(7rs + 9r) - (-rs + 3s)$

   d. $(-2z^2 + 6z - 3) + (2z^2 - 8z + 6)$

   e. $(-x^2 + 5x - 4) - (2x + 5)$

   f. $(6x^2 + 8x) - (2x + 3)$

Prepare

Determine the product.

1. $4x \cdot x$

2. $-3x \cdot 10x$

3. $-x(5 + 4x)$

4. $(2x - 8) \cdot (12x)$

2 Multiplying Polynomials

OBJECTIVES

- Model the multiplication of a binomial by a binomial using algebra tiles.
- Use multiplication tables to multiply binomials.
- Use the distributive property to multiply binomials.
- Recognize and use special products when multiplying binomials.

NEW KEY TERMS

- difference of two squares
- perfect square trinomial

You have added and subtracted polynomials.

How do you multiply polynomials?

Getting Started

Modeling Binomials

So far, you have learned how to add and subtract polynomials. But what about multiplying polynomials?

Let's consider the binomials $x + 1$ and $x + 2$. You can use algebra tiles to model the two binomials and determine their product.

WORKED EXAMPLE

Represent each binomial with algebra tiles.

x	1

$x + 1$

x	1	1

$x + 2$

Create an area model using each binomial.

•	x	1	1
x	x^2	x	x
1	x	1	1

1. What is the product of $(x + 1)(x + 2)$?

2. How would the model change if the binomial $x + 2$ was changed to $x + 4$? What is the new product of $x + 1$ and $x + 4$?

ACTIVITY 2.1

Multiplying with Algebra Tiles

In the Getting Started, you determined the product of $x + 1$ and $x + 2$.

1. Angel represented the product of $(x + 1)$ and $(x + 2)$ as shown.

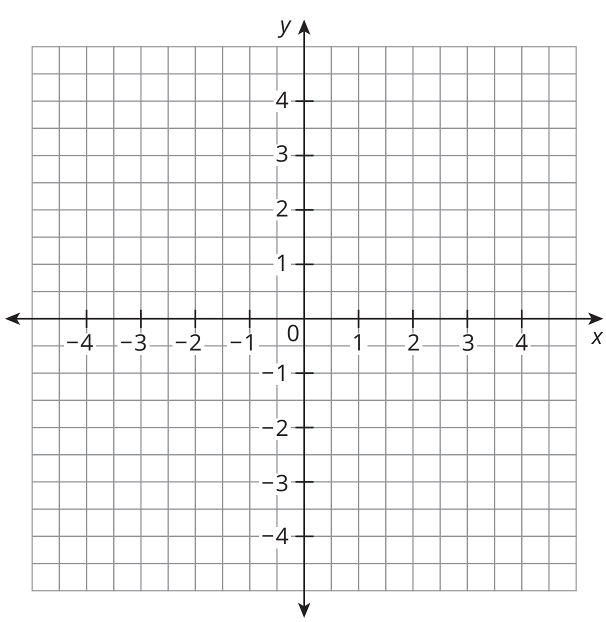
•	x	1
x	x^2	x
1	x	1
1	x	1

Gabriela told Angel that he incorrectly represented the area model because it does not look like the model in the Getting Started. Angel replied that it doesn't matter how the binomials are arranged in the model.

Who's Correct?

2. Use graphing technology to verify.
$(x + 1)(x + 2) = x^2 + 3x + 2$

a. Sketch both graphs on the coordinate plane.



b. How do the graphs verify that $(x + 1)(x + 2)$ and $x^2 + 3x + 2$ are equivalent?

c. Plot and label the x-intercepts and the y-intercept on your graph. How do the forms of each expression help you identify these points?

3. Use algebra tiles to determine the product of the binomials in each.

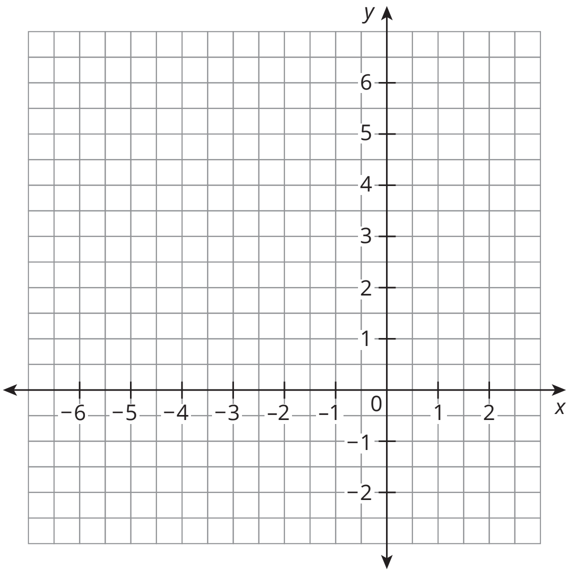
a. $x + 2$ and $x + 3$

b. $x + 2$ and $x + 4$

c. $2x + 3$ and $3x + 1$

4. Verify that the products you determined in Question 3 are correct using graphing technology. Write each pair of factors and the product. Then, sketch each graph on the coordinate plane.

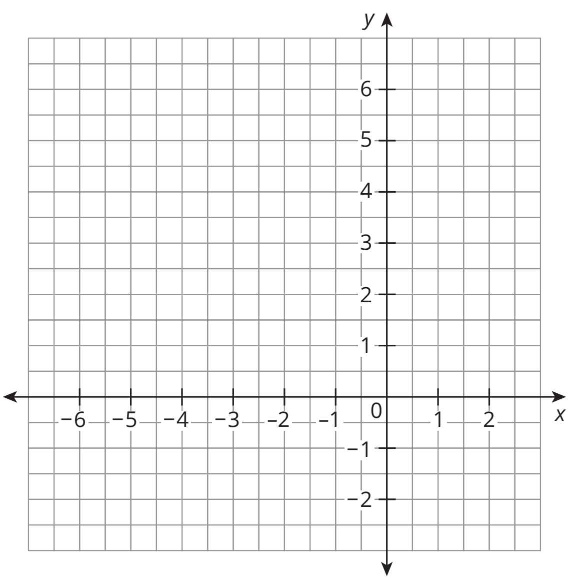
a.



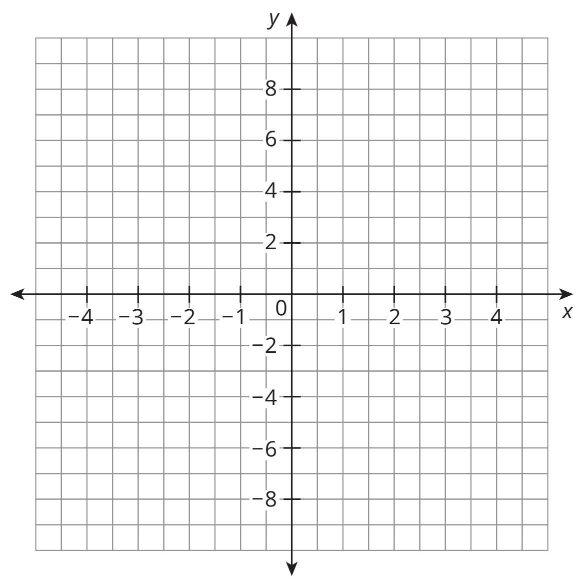
Think About . . .

How are the *x*-intercepts represented in the linear binomial expressions?

b.



c.



ACTIVITY 2.2

Using Area Models and the Distributive Property

While using algebra tiles is one way to determine the product of polynomials, they can also become difficult to use when the terms of the polynomials become more complex.

Luis was calculating the product of the binomials $4x + 7$ and $5x - 3$. He thought he didn't have enough algebra tiles to determine the product. Instead, he performed the calculation using the model shown.

1. Describe how Luis calculated the product of $4x + 7$ and $5x - 3$.

Luis

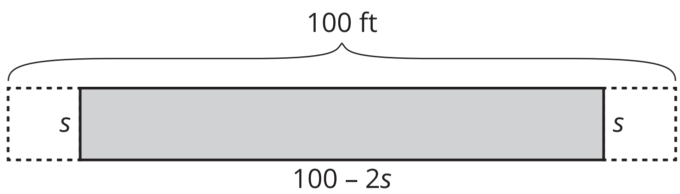
$\cdot$	$5x$	-3
$4x$	$20x^2$	$-12x$
7	$35x$	-21

$20x^2 + 23x - 21$

2. How is Luis's method similar to and different from using the algebra tiles method?

Luis used a multiplication table to calculate the product of the two binomials. By using a multiplication table, you can organize the terms of the binomials as factors of multiplication expressions. You can then use the distributive property of multiplication to multiply each term of the first polynomial with each term of the second polynomial.

Consider the dog enclosure scenario from the previous topic.



The area of the enclosure is expressed as $A(s) = s(100 - 2s)$, or the product of a monomial and a binomial.

3. Consider how Malik and Hannah wrote an equivalent polynomial function in standard form by calculating the product.

Malik

$\cdot$	100	$-2s$
s	$100s$	$-2s^2$

$A(s) = -2s^2 + 100s$

Hannah

$A(s) = s(100 - 2s)$

$A(s) = 100s - 2s^2$

$A(s) = -2s^2 + 100s$

Remember . . .
Develop a habit of writing answers in standard form. It makes them easier to compare with others' answers.

a. Describe the strategy Malik used to calculate the product.

b. How is Malik's strategy similar to Hannah's strategy?

Consider the Alamo Tours scenario from the previous topic. The revenue for the business is expressed as the product of a binomial times a binomial.

Revenue	=	**Number of Tours**	·	**Price per Tour**
↓		↓		↓
$r(x)$	=	$(10x + 100)$	·	$(50 - x)$

4. Finish Malik's process to write an equivalent polynomial function for revenue in standard form.

The process of using a multiplication table to multiply polynomials is referred to as an *area model*.

·	50	$-x$
$10x$	$500x$	$-10x^2$
100		

5. Use an area model to calculate the product of each polynomial. Write each product in standard form.

Ask Yourself . . .
Does it matter where you place the polynomials in the multiplication table?

a. $(3x + 2)(x - 4)$

b. $(x - 5)(x + 5)$

c. $(2x + 3)^2$

d. $(4x^2 + x - 1)(3x - 7)$

In Question 3, Hannah uses the distributive property to multiply a monomial and a binomial. She wants to use the distributive property to multiply any polynomials.

Think About . . .

How can you use technology to check your answers?

WORKED EXAMPLE

Consider the polynomials $x + 5$ and $x - 2$. You can use the distributive property to multiply these polynomials.

Distribute x to each term of $(x - 2)$ and then distribute 5 to each term of $(x - 2)$.

$$(x + 5)(x - 2) = (x)(x - 2) + (5)(x - 2)$$
$$= x^2 - 2x + 5x - 10$$
$$= x^2 + 3x - 10$$

6. Use the distributive property to determine each product. Write the polynomial in standard form.

a. $6(7x - 3)$

b. $5(3x^2 - 2x + 7)$

c. $(5x - 1)(2x + 1)$

d. $(x - 7)(x + 7)$

e. $(x^2 + 5x + 2)(4x + 2)$

f. $(2x^2 + 1)(3x^2 + x - 1)$

7. Explain the mistake in Ana's thinking. Then, determine the correct product.

Ana

$$(x + 4)^2 = x^2 + 16.$$

I can just square each term to determine the product.

ACTIVITY 2.3 Rewriting Quadratic Functions

You can use polynomial multiplication to rewrite quadratic functions written in factored form or vertex form.

WORKED EXAMPLE

Write the quadratic function $f(x) = (2x + 1)(x - 5)$ in standard form:

Step 1: Apply the distributive property. $f(x) = 2x(x - 5) + 1(x - 5)$

Step 2: Apply the distributive property. $f(x) = 2x^2 - 10x + x - 5$

Step 3: Combine like terms. $f(x) = 2x^2 - 9x - 5$

Write the quadratic function $g(x) = (x - 1)^2 + 6$ in standard form:

Step 1: Write the power as a multiplication. $g(x) = (x - 1)(x - 1) + 6$

Step 2: Apply the distributive property. $g(x) = x(x - 1) - 1\,(x - 1) + 6$

Step 3: Apply the distributive property. $g(x) = x^2 - x - x + 1 + 6$

Step 4: Combine like terms. $g(x) = x^2 - 2x + 7$

1. Write each quadratic equation in standard form.

a. $d(x) = (x - 4)(2x + 1)$

b. $p(x) = (4x + 2)(4x - 2)$

c. $k(x) = (2x - 3)^2 - 5$

d. $c(x) = (3x + 1)^2 + 4$

ACTIVITY 2.4

Special Products When Multiplying Binomials

In this activity, you will investigate the product of two linear factors when one is the sum of two terms and the other is the difference of the same two terms and when the two linear factors are the same.

1. Determine each product.

   a. $(x - 4)(x + 4) =$
   $(x + 4)(x + 4) =$
   $(x - 4)(x - 4) =$

   b. $(x - 3)(x + 3) =$
   $(x + 3)(x + 3) =$
   $(x - 3)(x - 3) =$

   c. $(3x - 1)(3x + 1) =$
   $(3x + 1)(3x + 1) =$
   $(3x - 1)(3x - 1) =$

   d. $(2x - 1)(2x + 1) =$
   $(2x + 1)(2x + 1) =$
   $(2x - 1)(2x - 1) =$

2. What patterns do you notice between the factors and the products?

3. Multiply each pair of binomials.

   a. $(ax - b)(ax + b) =$

   b. $(ax + b)(ax + b) =$

   c. $(ax - \mathrm{b})(ax - b) =$

In Questions 1 and 3, you should have observed a few special products. The first type of special product is called the *difference of two squares*. The **difference of two squares** is an expression in the form $a^2 - b^2$ that has factors $(a - b)(a + b)$.

4. List the expressions in Questions 1 and 3 that are examples of the difference of two squares.

The second type of special product is called a *perfect square trinomial*. A **perfect square trinomial** is an expression in the form $a^2 + 2ab + b^2$ or the form $a^2 - 2ab + b^2$. A perfect square trinomial can be written as the square of a binomial.

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

5. List the expressions in Questions 1 and 3 that are examples of perfect square trinomials.

Talk the Talk

Special Products

1. Use special products to determine each product.

   a. $(x - 8)(x - 8)$

   b. $(x + 8)(x - 8)$

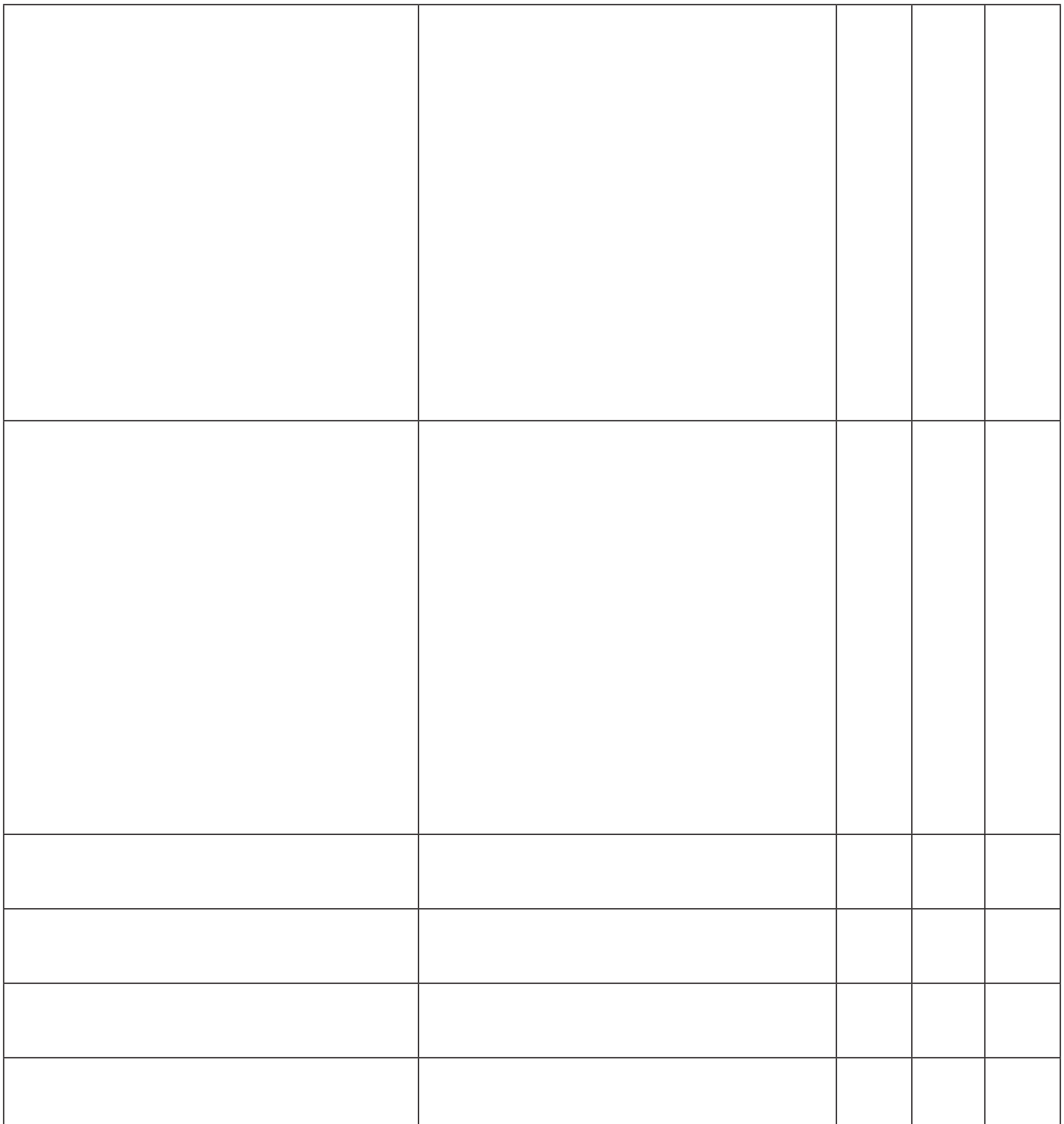
   c. $(x + 8)^2$

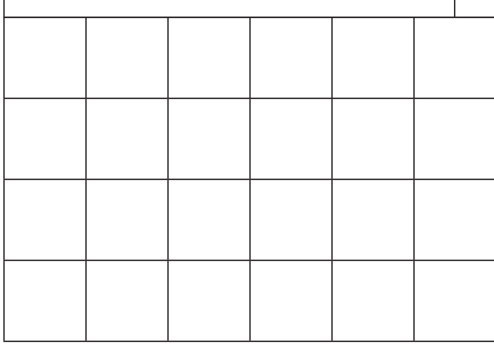
   d. $(3x + 2)^2$

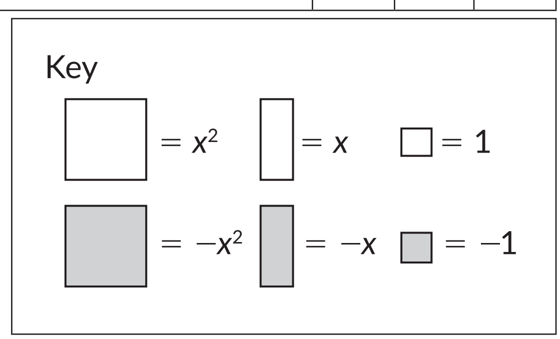
   e. $(3x - 2)(3x - 2)$

   f. $(3x - 2)(3x + 2)$

Algebra Tiles







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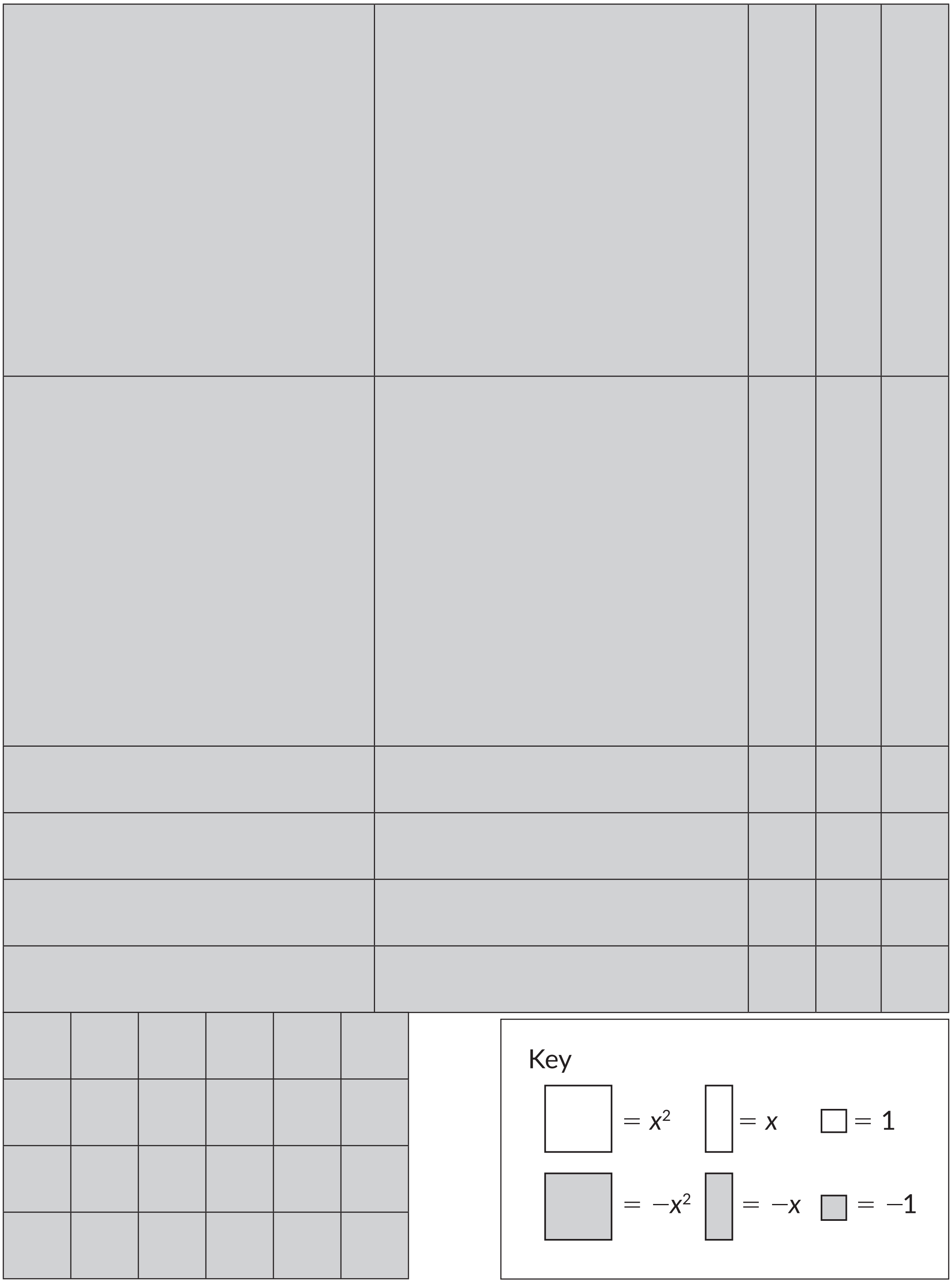
So you can cut out the Algebra Tiles on the other side

Algebra Tiles



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Lesson 2 Assignment

Write

In your own words, explain how to use the distributive property to multiply two binomials. Provide an example.

Remember

The difference of two squares is an expression in the form $a^2 - b^2$ that has factors $(a + b)(a - b)$.

A perfect square trinomial is an expression in the form $a^2 + 2ab + b^2$ or in the form $a^2 - 2ab + b^2$ that has the factors $(a + b)^2$ and $(a - b)^2$, respectively.

Practice

Use an area model to calculate each product.

1. $(4x - 1)(6 + 3x)$

·		

2. $(-5x + 5)^2$

·	$-5x$	5
$-5x$		
5		

3. $(x^2 + 3x - 2)(2x + 5)$

·		

Lesson 2 Assignment

4. $(x + 2)(x - 2)$

$\cdot$		

Use the distributive property to determine each product.
Write the polynomial in standard form.

5. $-3(4x - 5)$

6. $8(2x^2 - 4x + 3)$

7. $(x + 9)(x + 9)$

8. $(2x - 1)(x + 3)$

9. $(x + 4)(x^2 + 3x - 2)$

10. $(x^2 + 3)(3x^2 - 2x + 5)$

11. Rewrite the equation $y = (x - 3)^2 + 6$ in standard form.

Prepare

Determine each quotient.

1. $9x \div 3x$

2. $4x^2 \div 2x$

3. $x \div x$

3 Polynomial Division

OBJECTIVES

- Describe similarities between polynomials and integers.
- Determine factors of a polynomial using one or more roots of the polynomial.
- Compare polynomial long division to integer long division.
- Determine factors through polynomial long division.
- Use the remainder theorem to evaluate polynomial equations and functions.

NEW KEY TERMS

- factor theorem
- polynomial long division
- remainder theorem

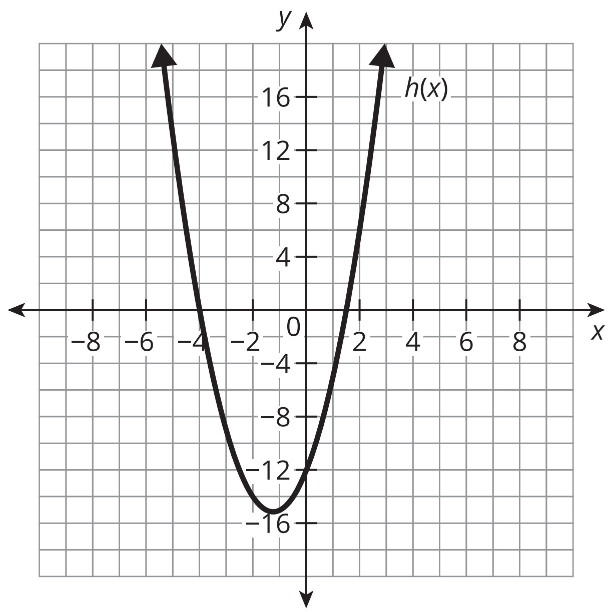
You know how to divide integers using the long division algorithm. How can you use a similar algorithm to divide polynomials?

Getting Started

The $x - r$ Factor

You have analyzed the graphs of polynomials to determine the type and location of the zeros. How can you determine the factors of a polynomial given its algebraic representation and one of its factors?

1. Analyze the graph of the function $h(x) = 2x^2 + 5x - 12$.



a. Describe the key characteristics of $h(x)$.

b. Describe the number and types of zeros of $h(x)$.

The **factor theorem** states that a polynomial function $p(x)$ has $x - r$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$.

You can use the factor theorem to show that a linear expression is a factor of a polynomial.

WORKED EXAMPLE

Consider the graph of the polynomial function $h(x) = 2x^2 + 5x - 12$ in Question 1.

The graph appears to have a zero at $(-4, 0)$, so a possible linear factor of the polynomial is $(x + 4)$.

Determine the value of the polynomial at $x = -4$, or $h(-4)$.

$$\begin{aligned} h(-4) &= 2(-4)^2 + 5(-4) - 12 \\ &= 32 - 20 - 12 \\ &= 0 \end{aligned}$$

So, $(x + 4)$ is a linear factor of the polynomial function.

2. Consider that $d(x) = (x + 4)$ and $d(x) \cdot q(x) = h(x)$.

   a. What do you know about the function $q(x)$?

   b. Can you write the algebraic representation for $q(x)$? Explain your reasoning.

ACTIVITY 3.1

Polynomial Long Division

To solve $0 = 2x^2 + 5x - 12$, you need to factor the polynomial and use the zero product property to determine its zeros.

> **Remember . . .**
> Recall that $a \div b$ is $\frac{a}{b}$, where $b \neq 0$.

The fundamental theorem of algebra states that every polynomial equation of degree n must have n roots. This means that every polynomial can be written as the product of n factors of the form $(ax + b)$. For example, $2x^2 - 3x - 9 = (2x + 3)(x - 3)$.

If 2 is a factor of 24, then 24 can be divided by 2 without a remainder. In the same way, the factors of a polynomial divide into that polynomial without a remainder.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.

WORKED EXAMPLE

Integer Long Division	Polynomial Long Division	
$3660 \div 12$ or $\frac{3660}{12}$ $$\begin{array}{r} 305 \\ 12\overline{)3660} \\ -36 \\ 6 \\ \underline{-0} \\ 60 \\ \underline{-60} \\ 0 \end{array}$$	$(2x^2 + 5x - 12) \div (x + 4)$ or $\frac{2x^2 + 5x - 12}{x + 4}$ $$\begin{array}{r} \text{Ⓐ}2x \;\; \text{Ⓓ}{-3} \\ x + 4\overline{)2x^2 + 5x - 12} \\ \text{Ⓑ}\underline{-(2x^2 + 8x)} \downarrow \;\text{Ⓒ} \\ -3x - 12 \\ \underline{-(-3x - 12)} \\ \text{Remainder } 0 \end{array}$$	A. Divide $\frac{2x^2}{x} = 2x$. B. Multiply $2x(x + 4)$ and then subtract. C. Bring down -12. D. Divide $\frac{-3x}{x} = -3$. E. Multiply $-3(x + 4)$ and then subtract.

1. Analyze the Worked Example that shows integer long division and polynomial long division.

   a. In what ways are the integer and polynomial long division algorithms similar?

   b. Rewrite each expression as a product of its factors.

      $3660 =$ ______ $\cdot$ ______ $2x^2 + 5x - 12 =$ ________ $\cdot$ ________

   c. Write the product of the linear factors of $2x^2 + 5x - 12$ as a function, $f(x)$.

   d. Determine the zeros of the function.

2. Determine whether $(x - 7)$ is a factor of $x^2 + 5x - 84$ using:

   a. Polynomial long division

   b. Factor theorem

   c. Which method do you prefer and why?

3. Use polynomial long division or the factor theorem to answer each question. Show your work and explain your reasoning.

   a. Is the expression $(x - 2)$ a factor of $12x^2 - 19x + 10$?

   b. Is the expression $(x + 18)$ a factor of $x^2 - 324$?

ACTIVITY 3.2

The Remainder Theorem

You learned that the process of dividing polynomials is similar to the process of dividing integers. Sometimes, when you divide two integers there is a remainder, and sometimes, there is not a remainder. What does each case mean? In this activity, you will investigate what the remainder means in terms of polynomial division.

1. Determine the quotient for each. Show all your work.

   a. $(5x - 12) \div (x - 4)$

   b. $(4x^2 + 31x + 21) \div (x + 7)$

   c. $(5x^2 - 23x + 17) \div (x - 2)$

   d. $(8x^2 + 3x + 6) \div (x^2 - x + 4)$

2. Consider Question 1 parts (a) through (d) to answer each question.

   a. When there is a remainder, is the divisor a factor of the dividend? Explain your reasoning.

   b. Describe the remainder when you divide a polynomial by one of its factors.

Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

For example, $\frac{283}{12} = 23 + \frac{7}{12} = 23\frac{7}{12}$.

You can verify your solution remembering that:

dividend = (divisor)(quotient) + remainder.

$$\begin{aligned} 283 &= 12(23) + 7 \\ 283 &= 276 + 7 \\ 283 &= 283 \end{aligned}$$

3. Rewrite your solutions from Question 1 in the form $\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$ and then verify each solution.

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear expression}} = \text{quotient} + \frac{\text{remainder}}{\text{linear expression}}$$

or

$$p(x) = (\text{linear expression})\ (\text{quotient}) + \text{remainder}.$$

Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R.

$$p(x) = (x - r) \cdot q(x) + \text{R}$$

4. Consider each dividend in Question 1 as a function, $p(x)$.

   a. In part (a), evaluate $p(x)$ for $x = 4$.

   b. In part (b), evaluate $p(x)$ for $x = -7$.

   c. In part (c), evaluate $p(x)$ for $x = 2$.

   d. What is the relationship between the remainder and the divisor in each problem?

Remember . . .

The factor theorem states that a polynomial has a linear polynomial as a factor if and only if the remainder is zero. Therefore, if R $= 0$, then $f(r) = 0$, and $(x - r)$ is a factor of $f(x)$.

5. What conclusion can you make about any polynomial evaluated at r?

The **remainder theorem** states that when any polynomial function, $f(x)$, is divided by a linear expression of the form $(x - r)$, the remainder $R = f(r)$, or the value of the function when $x = r$.

6. Use the remainder theorem to complete the table.

Dividend $p(x)$	Divisor $(x - r)$	r	Remainder, R $p(r)$
$4x^2 - 5x - 21$	$x - 3$		
$2x^2 + 3x - 49$	$x + 7$		

7. Determine the unknown in each.

a. $\frac{x}{7} = 18$ R 2. Determine x.

b. $\frac{p(x)}{x + 3} = 5x - 2$ R (-4). Determine the function $p(x)$.

c. Describe the similarities and differences in your solution strategies.

Talk the Talk

A Polynomial Divided

1. Given the information, determine whether each statement is true or false. Explain your reasoning.

$$p(x) = 3x^2 - 19x + 20, \text{ and}$$
$$p(x) \div (x + 6) = 3x - 1 + \frac{14}{x - 6}$$

a. $p(6) = 14$

b. $p(x) = (x + 6)(3x - 1) + 14$

c. $(x - 2)$ is a factor of $p(x)$

d. $(x - 5)$ is a factor of $p(x)$

2. Explain the difference between the remainder theorem and the factor theorem.

Lesson 3 Assignment

Write

Write an example for each term using the dividend $x^2 - 2x + 4$ and the divisor $(x + 1)$.

1. Factor theorem
2. Polynomial long division
3. Remainder theorem

Remember

A polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. When any polynomial equation or function $f(x)$ is divided by a linear expression of the form $(x - r)$, the remainder is $R = f(r)$, or the value of the function when $x = r$.

Practice

1. Use the factor theorem to determine whether each linear expression is a factor of $2x^2 - 29x - 48$.

   a. $x + 3$

   b. $x - 16$

   c. $x + \frac{3}{2}$

Lesson 3 Assignment

2. Determine the quotient for each problem. Show all your work.

a. $\frac{2x^2 + 6x - 7}{x + 1}$

b. $(3x^2 - \frac{5}{2}x - 2) \div (x + \frac{1}{2})$

c. $\frac{14x^2 + 10x - 1}{x - 2}$

d. $(3x^2 + 2x - 5) \div (x^2 - 4)$

Lesson 3 Assignment

3. The table of values represents the function $f(x) = x^2 + 10x - 96$.

a. Determine one of the factors of $f(x)$ without using a calculator. Explain your reasoning.

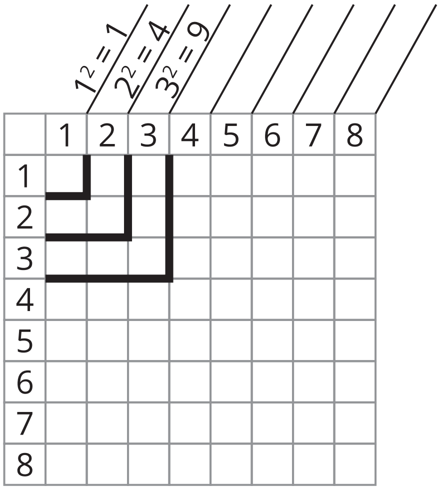
(x)	-18	-12	0	6	8
$f(x)$	48	-72	-96	0	48

b. Completely factor $f(x)$ without using a calculator.

c. Determine all of the zeros of $f(x)$ without using a calculator.

Prepare

1. Complete the grid by continuing to make squares with side lengths of 4 through 8. Connect the side lengths and then write an equation using exponents to represent each perfect square.



 Name: ____________________

Polynomial Operations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Polynomial Operations* topic by:

TOPIC 2: *Polynomial Operations*	Beginning of Topic	Middle of Topic	End of Topic
classifying a polynomial as a monomial, binomial, or trinomial based on the number of terms and determine the degree of the polynomial.	○	○	○
adding and subtracting polynomials.	○	○	○
multiplying polynomials.	○	○	○
dividing polynomials.	○	○	○
recognizing special products, including the difference of two squares, and using them to multiply polynomials.	○	○	○

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Polynomial Operations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. What concepts are you working to understand from this topic? How will you work to build your understanding of these concepts?

TOPIC 2 SUMMARY

Polynomial Operations

LESSON 1 Adding and Subtracting Polynomials

Linear and quadratic expressions are part of a larger group of expressions known as *polynomials*. A **polynomial** is an expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k , where a, called the *coefficient*, is a real number and k is a non-negative integer. A polynomial is written in standard form when the terms are in descending order, starting with the term with the greatest degree and ending with the term with the least degree.

$$a_1x^k + a_2x^{k-1} + \dots a_nx^0$$

Each product in a polynomial is called a *term*. Polynomials are named according to the number of terms: a **monomial** has exactly 1 term, a **binomial** has exactly 2 terms, and a **trinomial** has exactly 3 terms.

The exponent of a term is the degree of the term, and the greatest exponent in a polynomial is the **degree of the polynomial**.

The characteristics of the polynomial $13x^3 + 5x + 9$ are shown in the chart.

	1st term	2nd term	3rd term
Term	$13x^3$	$5x$	9
Coefficient	13	5	9
Power	x^3	x^1	x^0
Exponent	3	1	0

This trinomial has a degree of 3 because 3 is the greatest degree of the terms in the trinomial.

Polynomials can be added or subtracted by identifying the like terms of the polynomial functions, using the associative property to group the like terms together and combining the like terms to simplify the expression.

For example, to add the polynomial expressions $(7x^2 - 2x + 12)$ and $(8x^3 + 2x^2 - 3x)$, use the associative property to combine like terms.

$(7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)$

$8x^3 + (7x^2 + 2x^2) + (-2x - 3x) + 12$

$8x^3 + 9x^2 - 5x + 12$

NEW KEY TERMS

- polynomial [polinomio]
- monomial [monomio]
- binomial [binomio]
- trinomial [trinomio]
- degree of a polynomial [grado de un polinomio]
- difference of two squares
- perfect square trinomial
- factor theorem [teorema del factor]
- polynomial long division [división (larga) de polinomios]
- remainder theorem

LESSON 2

Multiplying Polynomials

The product of 2 binomials can be determined by using an area model with algebra tiles. Another way to model the product of 2 binomials is a multiplication table, which organizes the two terms of the binomials as factors of multiplication expressions.

Example 1

$(2x + 1)(x + 3)$

$\cdot$	x	1	1	1
x	x^2	x	x	x
x	x^2	x	x	x
1	x	1	1	1

$(2x + 1)(x + 3) = 2x^2 + 7x + 3$

Example 2

$(9x - 1)(5x + 7)$

$\cdot$	$9x$	-1
$5x$	$45x^2$	$-5x$
7	$63x$	-7

$$(9x - 1)(5x + 7) = 45x^2 - 5x + 63x - 7$$
$$= 45x^2 + 58x - 7$$

You can also use the distributive property to multiply polynomials.

To multiply $4(x - 3)$, use the distributive property once.

$4(x - 3) = 4(x) - 4(3) = 4x - 12$

Depending on the number of terms in the polynomials, you may use the distributive property multiple times.

For example, to multiply the polynomials $x + 5$ and $x - 2$, first, use the distributive property to multiply each term of $x + 5$ by the entire binomial $x - 2$.

$$(x + 5)(x - 2) = (x)(x - 2) + (5)(x - 2)$$

Next, distribute x to each term of $x - 2$ and distribute 5 to each term of $x - 2$.

$$x^2 - 2x + 5x - 10$$

Finally, collect the like terms and write the solution in standard form.

$$x^2 + 3x - 10$$

You can use the same process to multiply any polynomials

For example:

$(x^2 + 4x - 7)(x^2 - 3x + 1)$

$= x^2 (x^2 - 3x + 1) + 4x(x^2 - 3x + 1) - 7(x^2 - 3x + 1)$

$= (x^4 - 3x^3 + x^2) + (4x^3 - 12x^2 + 4x) + (-7x^2 + 21x - 7)$

$= x^4 + x^3 - 18x^2 + 25x - 7$

There are special products of degree 2 that have certain characteristics. The expression in the form $a^2 - b^2$ that has factors $(a + b)(a - b)$ is called the

difference of two squares. A **perfect square trinomial** is formed by multiplying a binomial by itself. It is an expression in the form $a^2 + 2ab + b^2$, or in the form $a^2 - 2ab + b^2$. A perfect square trinomial can be written as the square of a binomial. In these cases, the factors are $(a + b)^2$ and $(a - b)^2$, respectively.

You can use what you know about multiplying polynomials to rewrite quadratic functions in different forms.

Write the quadratic function $f(x) = (2x + 1)(x - 5)$ in standard form:

Step 1: Apply the distributive property. $f(x) = 2x(x - 5) + 1(x - 5)$

Step 2: Apply the distributive property. $f(x) = 2x^2 - 10x + x - 5$

Step 3: Combine like terms. $f(x) = 2x^2 - 9x - 5$

Write the quadratic function $g(x) = (x - 1)^2 + 6$ in standard form:

Step 1: Write the power as a multiplication. $g(x) = (x - 1)(x - 1) + 6$

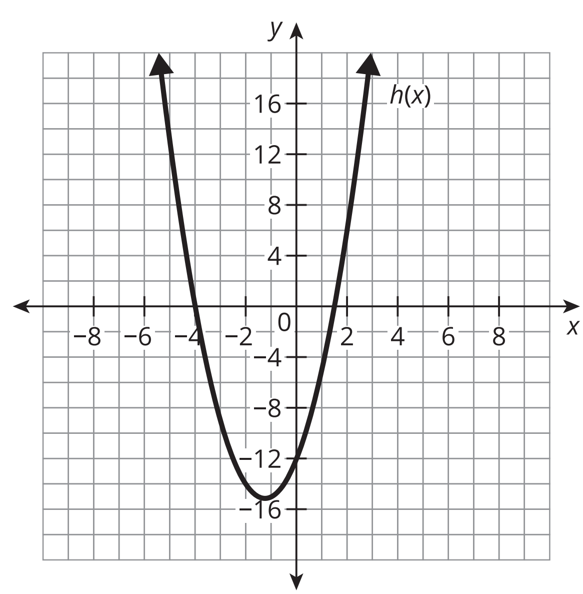
Step 2: Apply the distributive property. $g(x) = x(x - 1) - 1\,(x - 1) + 6$

Step 3: Apply the distributive property. $g(x) = x^2 - x - x + 1 + 6$

Step 4: Combine like terms. $g(x) = x^2 - 2x + 7$

LESSON 3 Polynomial Division

The **factor theorem** states that a polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. You can use the factor theorem to show that a linear expression is a factor of a polynomial.



Consider the graph of the polynomial function $h(x) = 2x^2 + 5x - 12$.

The graph appears to have a zero at (−4, 0), so a possible linear factor of the polynomial is $(x - (-4))$, or $(x + 4)$.

Determine the value of the polynomial at $x = -4$, or $h(-4)$.

$$h(-4) = 2(-4)^2 + 5(-4) - 12$$

$$h(-4) = 32 - 20 - 12$$

$$h(-4) = 0$$

So, $(x + 4)$ is a linear factor of the polynomial function.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division. For example, when 2 is a factor of 24, then 24 can be divided by 2 without a remainder. In the same way, the factors of a polynomial divide into that polynomial without a remainder. Therefore, you can use polynomial long division to determine whether a linear expression is a factor of a polynomial.

Integer Long Division	Polynomial Long Division
$3660 \div 12$ or $\frac{3660}{12}$ $\begin{array}{r} 305 \\ 12\overline{)3660} \\ -36 \\ \hline 6 \\ -0 \\ \hline 60 \\ -60 \\ \hline 0 \end{array}$	$(2x^2 + 5x - 12) \div (x + 4)$ or $\frac{2x^2 + 5x - 12}{x + 4}$ $\begin{array}{r} \text{Ⓐ}\,2x - \text{Ⓓ}\,3 \\ x + 4\overline{)2x^2 + 5x - 12} \\ \text{Ⓑ}\,-(2x^2 + 8x)\downarrow\quad\text{Ⓒ} \\ \hline -3x - 12 \\ -(-3x - 12) \\ \hline \text{Remainder } 0 \end{array}$ A. Divide $\frac{2x^2}{x} = 2x$. B. Multiply $2x(x + 4)$ and then subtract. C. Bring down -12. D. Divide $\frac{-3x}{x} = -3$. E. Multiply $-3(x + 4)$ and then subtract.

For the polynomial long division example above, the remainder is 0, which means that $(x + 4)$ is a factor of $2x^2 + 5x - 12$. The quotient is $(2x - 3)$, which is the other factor of the polynomial. Therefore, $2x^2 + 5x - 12 = (x + 4)(2x - 3)$.

Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear expression}} = \text{quotient} + \frac{\text{remainder}}{\text{linear expression}}$$

or

$p(x) = (\text{linear expression})(\text{quotient}) + \text{remainder}.$

Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R.

$p(x) = (x - r)q(x) + R$

Consider this example. Determine the quotient of $(3x^2 + 10x + 1) \div (x + 4)$.

$$
\begin{array}{r}
3x - 2 \\
x + 4 \overline{)3x^2 + 10x + 1} \\
-(3x^2 + 12x) \\
\hline
-2x + 1 \\
-(-2x - 8) \\
\hline
9
\end{array}
$$

The quotient of $(3x^2 + 10x + 1) \div (x + 4)$ is $3x - 2 + \frac{9}{x + 4}$.

$$\frac{3x^2 + 10x + 1}{x + 4} = 3x - 2 + \frac{9}{x + 4}$$

$$3x^2 + 10x + 1 = (x + 4)(3x - 2) + 9$$

The **remainder theorem** states that when any polynomial function, $f(x)$, is divided by a linear expression of the form $(x - r)$, the remainder $R = f(r)$, or the value of the function when $x = r$. Remember, the factor theorem states that a polynomial has a linear expression as a factor if and only if the remainder is zero. Therefore, if $R = 0$, then $f(r) = 0$ and $(x - r)$ is a factor of $f(x)$.

In the previous example, using polynomial long division we determined $R = 9$; therefore, by the factor theorem, $(x + 4)$ is not a factor of $3x^2 + 10x + 1$. This can be confirmed using the remainder theorem.

When $f(x) = 3x^2 + 10x + 1$, to determine if $(x + 4)$ is a factor of $f(x)$, we must determine the value of $f(-4)$.

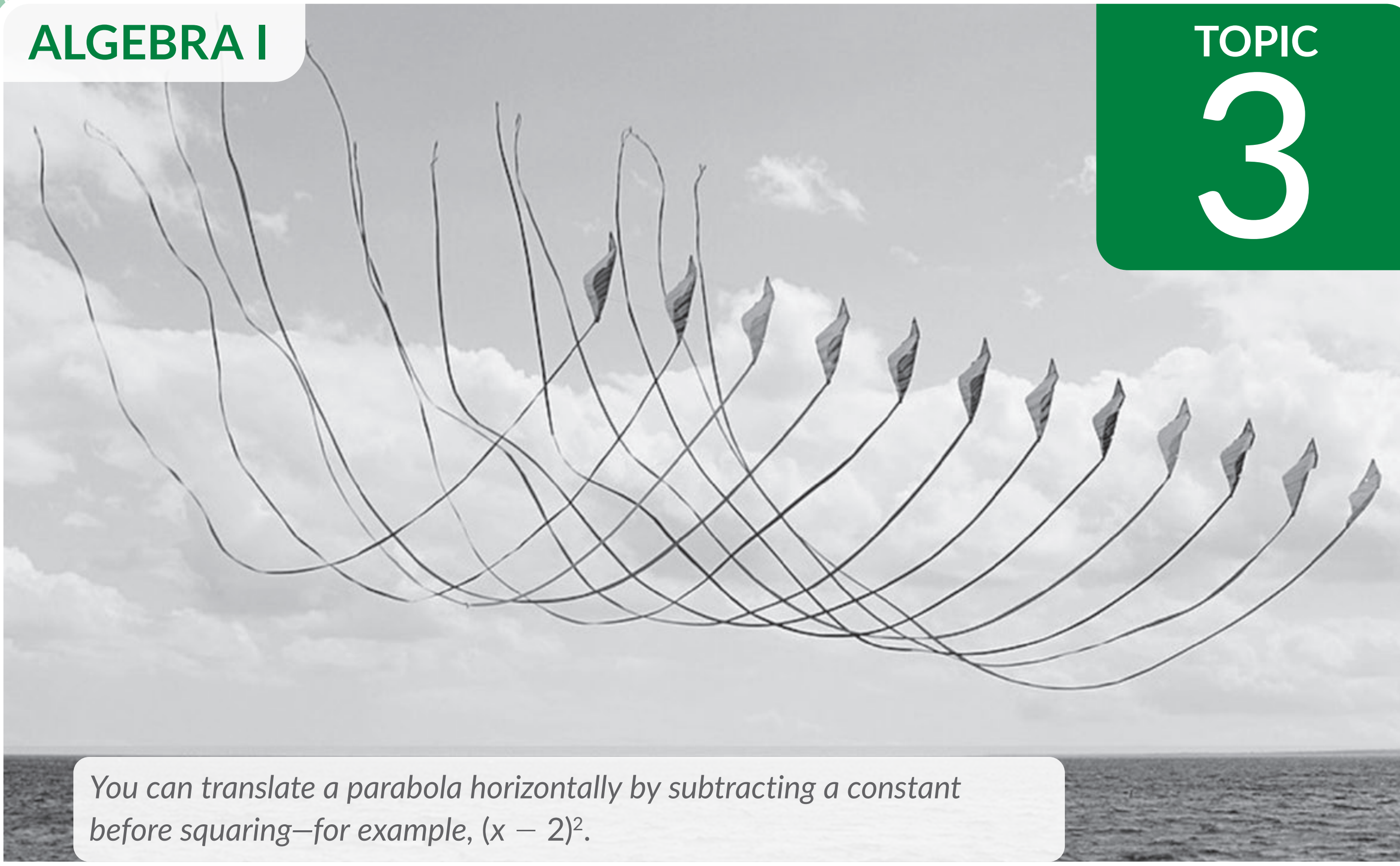
$$f(-4) = 3(-4)^2 + 10(-4) + 1$$
$$f(-4) = 9$$

This means the remainder of $f(x) = 9$, not 0; therefore, $(x + 4)$ is not a factor of $f(x) = 3x^2 + 10x + 1$.

You can also divide more complicated polynomials. For example, consider $(8x^2 + 3x + 6) \div (x^2 - x + 4)$. The process is similar.

$$
\begin{array}{r}
8 \quad R\,(11x - 26) \\
x^2 - x + 4 \overline{)8x^2 + 3x + 6} \\
-(8x \quad - 8x + 32) \\
\hline
11x \quad - 26 \\
8 + \frac{11x - 26}{x^2 - x + 4}
\end{array}
$$

TOPIC 3



You can translate a parabola horizontally by subtracting a constant before squaring—for example, $(x - 2)^2$.

Solving Quadratic Equations

LESSON 1 Representing Solutions to Quadratic Equations . . . **827**

LESSON 2 Solutions to Quadratic Equations in Vertex Form . . **843**

LESSON 3 Factoring and Completing the Square **855**

LESSON 4 The Quadratic Formula **879**

LESSON 5 Using Quadratic Functions to Model Data **901**

1 Representing Solutions to Quadratic Equations

OBJECTIVES

- Identify the zeros of a quadratic function, the roots of a quadratic equation, and the *x*-intercepts of a parabola using the equation of a quadratic function.
- Identify the double root of a quadratic equation as the two solutions of a quadratic equation at the minimum or maximum of the function.
- Write solutions of quadratic equations at specific output values using the axis of symmetry and the positive and negative square roots of the output value.
- Identify quadratic equations written as the difference of two perfect squares and rewrite these equations in factored form with a leading coefficient of 1.

NEW KEY TERMS

- principal square root
- roots
- double root
- zero product property

You have studied the graphs and equations for quadratic functions. How can you determine solutions of quadratic equations given different output values?

How can you use the structure of a parabola to understand the solutions of a quadratic equation?

Getting Started

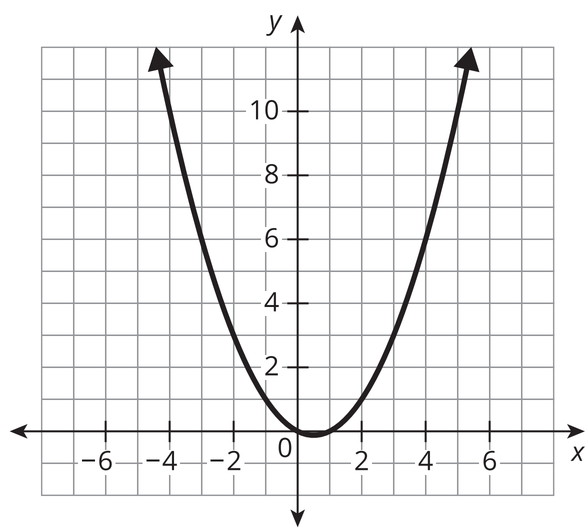
Shaking Hands

Consider one of the quadratic models you investigated in a previous topic.

Suppose that there is a monthly meeting at CIA headquarters for all employees. How many handshakes will it take for every employee at the meeting to shake the hand of every other employee at the meeting once?

You can model The Handshake Problem with the equation $f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$ or $f(n) = \frac{1}{2}(n)(n - 1)$.

1. Determine the equation for the axis of symmetry. Indicate the axis of symmetry on the graph.



2. Draw a horizontal line at $y = 6$.

   a. Where does the line $y = 6$ intersect the graph of $f(n)$?

   b. What is the distance of each point from the axis of symmetry?

c. What do the points of intersection mean in terms of this situation?

3. For each y, how many solutions does the equation $y = \frac{1}{2}n^2 - \frac{1}{2}n$ have?

4. Draw a horizontal line at $y = 0$ to determine the solutions to the equation $0 = \frac{1}{2}n^2 - \frac{1}{2}n$. How far is each solution away from the axis of symmetry?

ACTIVITY 1.1

Solutions of a Quadratic Function

Remember . . .

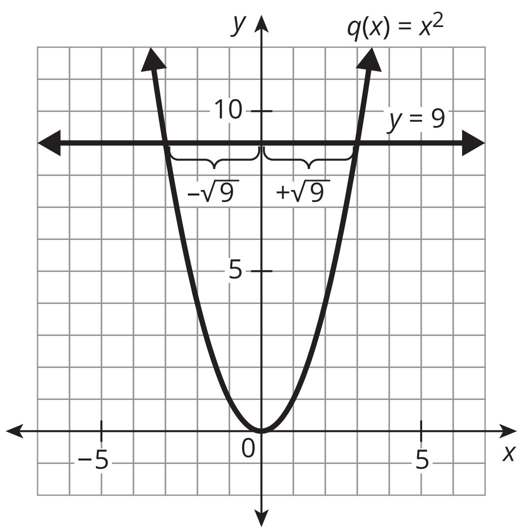
The square root property is $\sqrt{a^2} = \pm a$.

Recall that a quadratic function is a function of degree 2 because the greatest power for any of its terms is 2. This means that it has 2 zeros, or 2 solutions, at $y = 0$.

The two solutions of a basic quadratic function can be represented as square roots of numbers. Every positive number has two square roots, a positive square root (which is also called the **principal square root**) and a negative square root. To solve the equation $x^2 = 9$, you can take the square root of both sides of the equation.

$$\sqrt{x^2} = \pm\sqrt{9}$$
$$x = \pm 3$$

Solving $x^2 = 9$ on a graph means that you are looking for the points of intersection between $y = x^2$ and $y = 9$.



1. Consider the graph of the function $q(x) = x^2$ above.

   a. What is the equation for the axis of symmetry? Explain how you can use the function equation to determine your answer.

b. Explain how the graph shows the two solutions for the function at $y = 9$ and their relationship to the axis of symmetry. Use the graph and the function equation to explain your answer.

c. Describe how you can determine the two solutions for the function at $y = 2$. Indicate the solutions on the graph.

d. Describe how you can determine the two solutions for the function at each y-value for $y \geq 0$.

The quadratic function $q(x) = x^2$ has two solutions at $y = 0$. Therefore, it has 2 zeros: $x = +\sqrt{0}$ and $x = -\sqrt{0}$. These two zeros of the function, or roots of the equation, are the same number, 0, so $y = x^2$ is said to have a **double root,** or **1 unique root**.

The root of an equation indicates where the graph of the equation crosses the x-axis. A double root occurs when the graph just touches the x-axis but does not cross it.

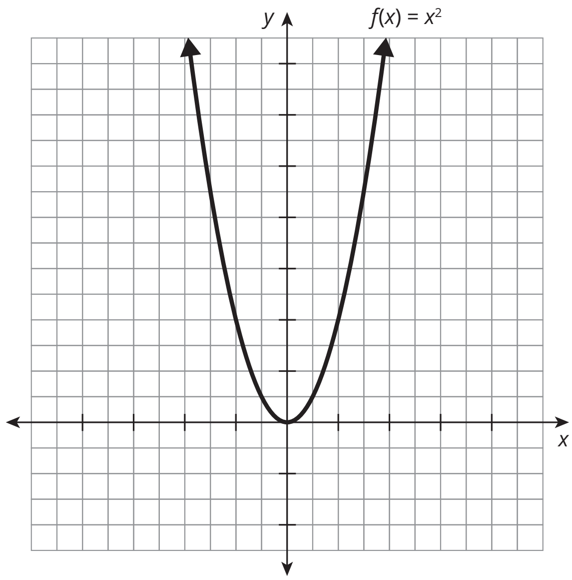
The x-coordinates of the x-intercepts of a graph of a quadratic function are called the **zeros** of the quadratic function. The zeros are called the **roots** of the quadratic equation.

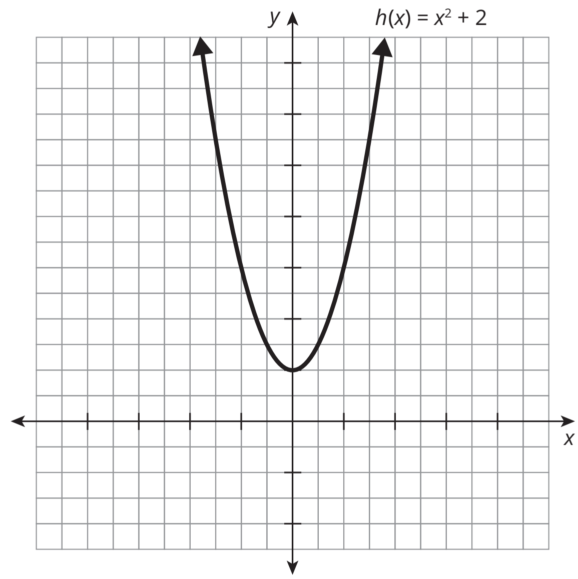
2. Use the graph and the function to explain why Fernando's equation is correct.

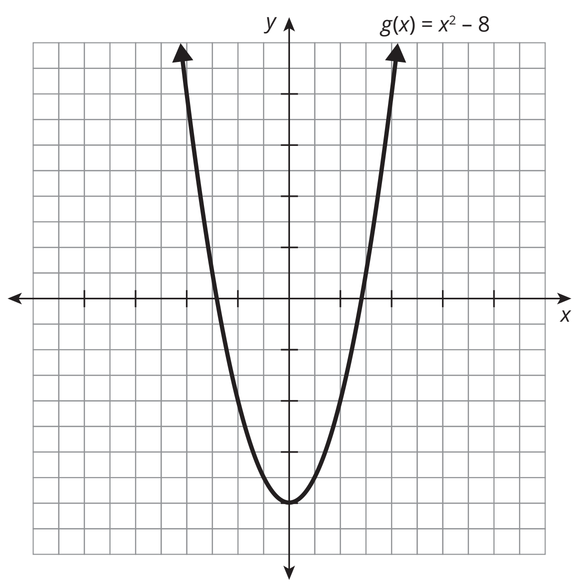
Fernando

The quadratic equation $y = x^2$ is symmetric about its axis of symmetry, $x = 0$. So, I can write the solution to the quadratic equation $y = x^2$, as $x = 0 \pm \sqrt{y}$.

The graphs of three quadratic functions, $f(x)$, $h(x)$, and $g(x)$, are shown.







3. Use the graphs to identify the solutions to each equation. Then, determine the solutions algebraically and write the solutions in terms of their respective distances from the axis of symmetry.

a. $14 = x^2 + 2$

b. $x^2 = 10$

c. $-5 = x^2 - 8$

d. $19 = x^2 + 4$

e. $x^2 - 8 = 1$

f. $6 = x^2$

4. Consider the graphs of $f(x)$, $h(x)$, and $g(x)$. Which function has a double root? Explain your answer.

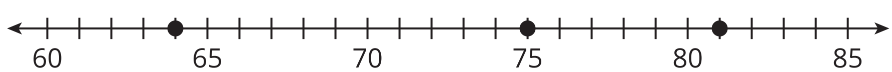
When you are solving quadratic equations, you may encounter solutions that are not perfect squares. You can either determine the approximate value of the radical or rewrite it in an equivalent radical form.

WORKED EXAMPLE

You can determine the approximate value of $\sqrt{75}$.

Determine the perfect square that is closest to, but less than, 75. Then, determine the perfect square that is closest to, but greater than, 75.

$$64 \leq 75 \leq 81$$



Determine the square roots of the perfect squares.

$$\sqrt{64} = 8 \qquad \sqrt{75} = ? \qquad \sqrt{81} = 9$$

Now that you know that $\sqrt{75}$ is between 8 and 9, you can test the squares of numbers between 8 and 9.

$$8.6^2 = 73.96 \qquad 8.7^2 = 75.69$$

Since 75 is closer to 75.69 than 73.96, 8.7 is the approximate square root of $\sqrt{75}$.

Ask Yourself . . .
Can you name all the perfect squares from 1^2 through 15^2?

WORKED EXAMPLE

You can use prime factors to rewrite $\sqrt{75}$ in an equivalent radical form.

First, rewrite the product of 75 to include any perfect square factors and then extract the square roots of those perfect squares.

$$\begin{aligned}\sqrt{75} &= \sqrt{3 \cdot 5 \cdot 5} \\ &= \sqrt{3 \cdot 5^2} \\ &= \sqrt{3} \cdot \sqrt{5^2} \\ &= 5\sqrt{3}\end{aligned}$$

Think About . . .
How could listing the prime factors of a radical expression help to extract square roots of perfect squares?

5. Estimate the value of each radical expression. Then, rewrite each radical by extracting all perfect squares, when possible.

a. $\sqrt{20}$

b. $\sqrt{18}$

c. $\sqrt{26}$

d. $\sqrt{116}$

6. Rewrite your answers from Question 3 by extracting perfect squares, when possible. Verify your rewritten answers using the graphs in Question 3.

ACTIVITY
1.2

Solutions from Standard Form to Factored Form

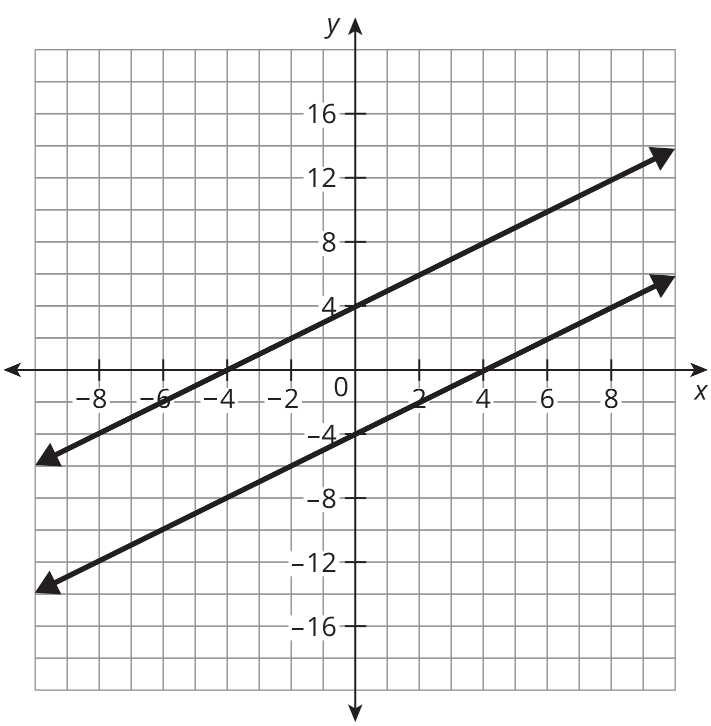
Recall that a quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$. In factored form, r_1 and r_2 represent the x-intercepts of the graph of the function.

Think About . . .
Do you recognize the form of this quadratic?

1. Determine the zeros of the function $z(x) = x^2 - 16$. Then, write the function in factored form.

The function $z(x)$ in factored form is a quadratic function made up of two linear factors. Let's analyze the linear factors as separate linear functions, $g(x)$ and $h(x)$. Therefore, $z(x) = g(x) \cdot h(x)$.

2. Complete the table by writing the algebraic expressions to represent $g(x)$ and $h(x)$ and then determine the output values for the two linear factors and the quadratic product. Finally, sketch a graph of $z(x)$.



x	$g(x)$	$h(x)$	$z(x)$
			$x^2 - 16$
−4			
−2			
0			
2			
4			

The **zero product property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

WORKED EXAMPLE

You can use the zero product property to identify the zeros of a function when the function is written in factored form.

$0 = x^2 - 16$

$0 = (x + 4)(x - 4)$	Rewrite the quadratic as linear factors.
$x - 4 = 0$ and $x + 4 = 0$	Apply the zero product property.
$x = 4$ $\quad$ $x = -4$	Solve each equation for *x*.

3. Explain how the zeros of the linear function factors are related to the zeros of the quadratic function product.

The function $z(x) = x^2 - 16$ has an *a*-value of 1 and a *b*-value of 0. You can use a similar strategy to determine the zeros of a function when the leading coefficient is not 1, but the *b*-value is still 0.

WORKED EXAMPLE

You can determine the zeros of the function $f(x) = 9x^2 - 1$ by setting $f(x) = 0$ and using the properties of equality to solve for *x*.

$$9x^2 - 1 = 0$$

$$9x^2 = 1$$

$$x^2 = \frac{1}{9}$$

$$x = \pm\frac{1}{3}$$

You can then use the leading coefficient of 9 and the zeros at $\frac{1}{3}$ and $-\frac{1}{3}$ to rewrite the quadratic function in factored form.

$$f(x) = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$$

4. Consider the Worked Example.

   a. Explain why $\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$.

   b. Use graphing technology to verify that $9x^2 - 1 = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$.
   How can you tell from the graph that the two equations are equivalent?

Three students tried to rewrite the quadratic function $f(x) = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$ as two linear factors using what they know about the difference of two squares.

Alexander

$9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) =$

$(9x - 3)\left(x + \frac{1}{3}\right)$

Jacob

$9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) =$

$(4.5x - 1.5)(4.5x + 1.5)$

Olivia

$9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) =$

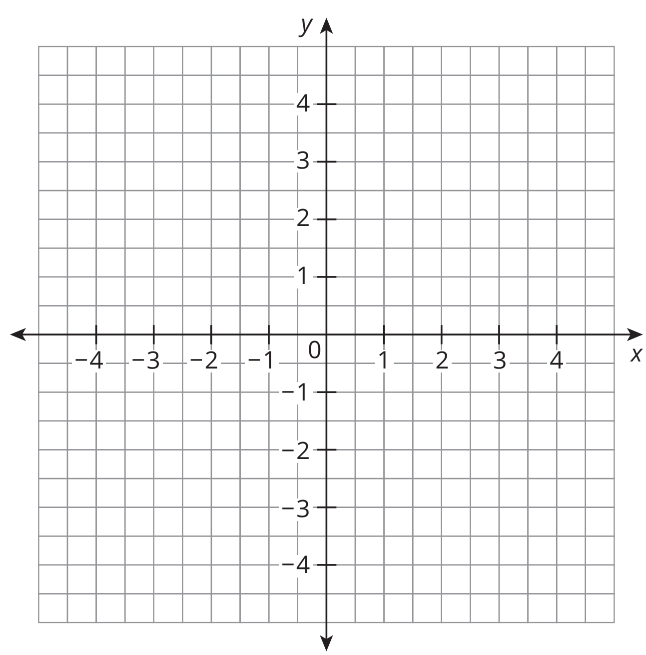
$(3x - 1)(3x + 1)$

5. Explain why Alexander and Jacob are incorrect and why Olivia is correct.

Ask Yourself . . .
Are these expressions still in factored form?

6. The graph of $f(x) = 9x^2 - 1$ is shown.

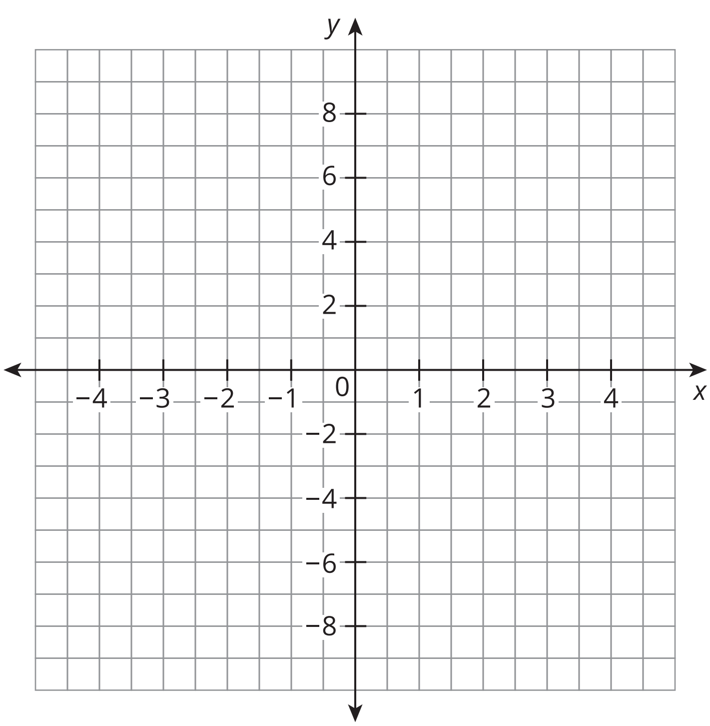
   a. Use Olivia's function, $f(x) = (3x - 1)(3x + 1)$, to sketch a graph of the linear factors. Then, use graphing technology to verify that $9x^2 - 1 = (3x - 1)(3x + 1)$.



b. What is the relationship of the zeros of the function and its two linear factors?

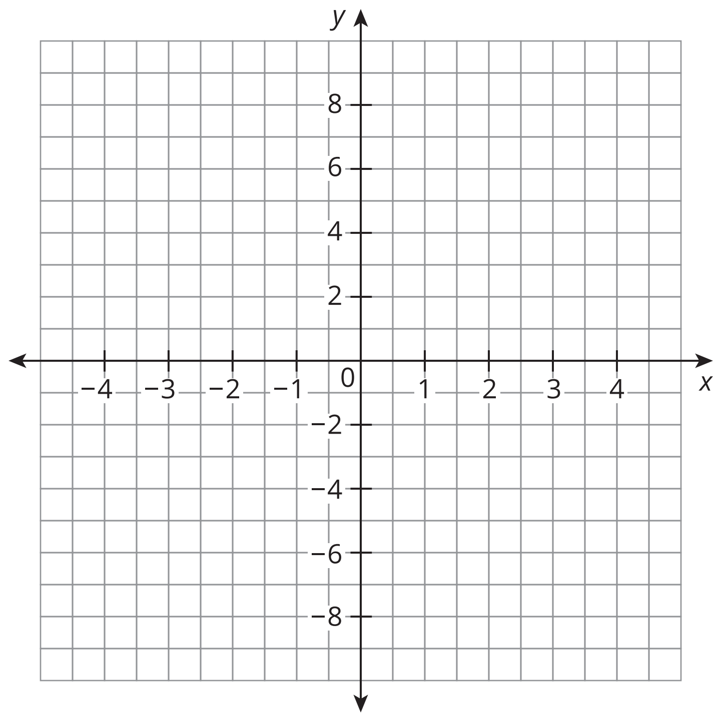
7. For each function:

- Sketch a graph. Label the axis of symmetry and the vertex.
- Use the properties of equality to identify the zeros and then write the zeros in terms of their respective distances from the line of symmetry.
- Use what you know about the difference of two squares to rewrite each quadratic as the product of two linear factors. Then, use the zero product property to verify the values of x, when $f(x) = 0$.
- Use graphing technology to verify that the product of the two linear factors is equivalent to the given function.

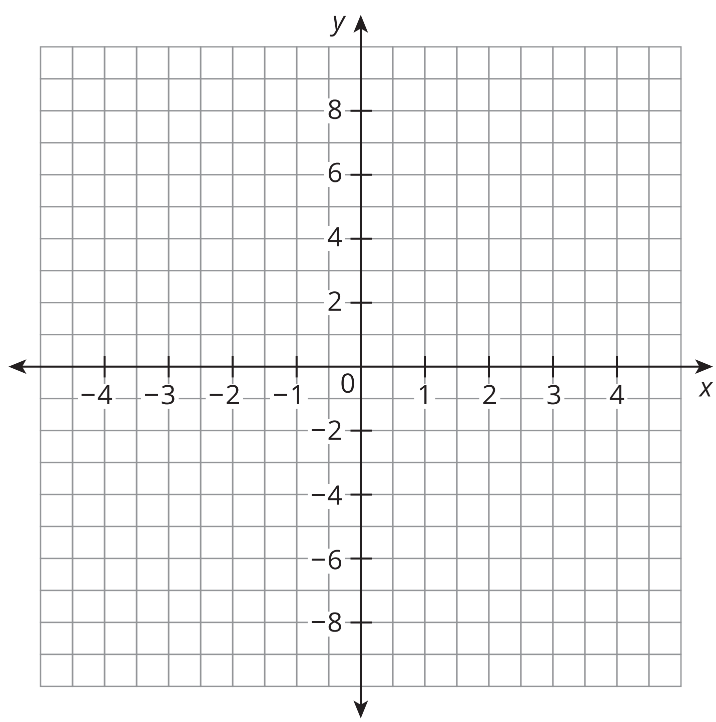


a. $f(x) = 4x^2 - 9$

b. $f(x) = x^2 - 2$



c. $f(x) = 25x^2 - 1$



Talk the Talk

The Difference of Squares

In this lesson you determined the zeros of quadratics written in the form $f(x) = ax^2 - c$.

1. Solve each equation.

   a. $x^2 - 25 = 0$

   b. $4x^2 - 1 = 0$

   c. $9x^2 - 2 = 0$

   d. $x^2 - 80 = 0$

2. Rewrite each quadratic function as two linear factors using what you know about the difference of two squares.

   a. $f(x) = x^2 - 49$

   b. $f(x) = \frac{4}{9}x^2 - 1$

   c. $f(x) = 16x^2 - 10$

   d. $f(x) = x^2 + 9$

3. Explain how to write any function of the form $f(x) = ax^2 - c$, where a and c are any real numbers, as two linear factors using what you know about the difference of two squares.

Lesson 1 Assignment

Write

Complete each definition.

1. The zero product property states that if the product of two or more factors is equal to ________, then at least one factor must be equal to ________.

2. Every positive number has both a ________ square root and a ________ square root.

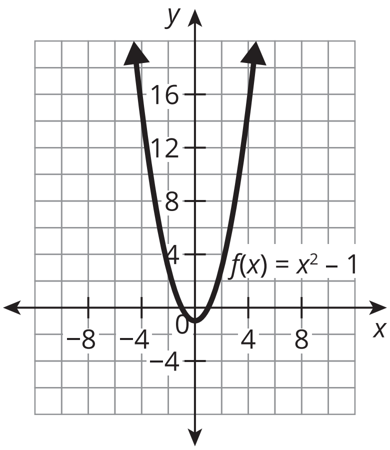
3. The function $f(x) = x^2$ has a __________ at (0, 0).

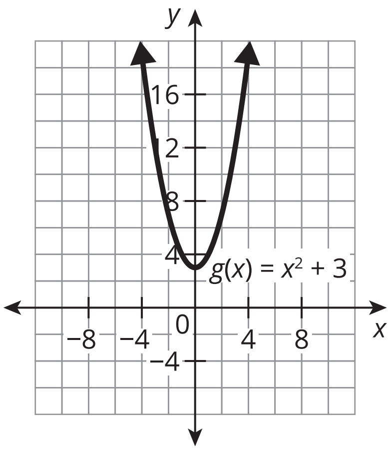
Remember

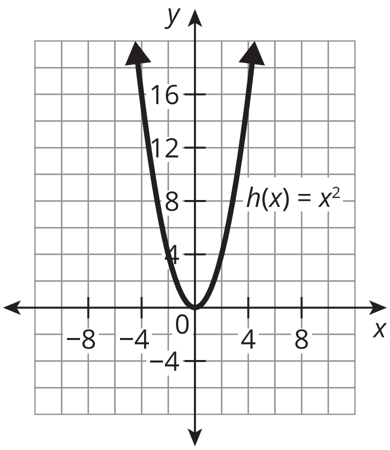
Any quadratic function of the form $f(x) = ax^2 - d$ can be rewritten as two linear factors in the form $(\sqrt{a}x - \sqrt{d})(\sqrt{a}x + \sqrt{d})$.

Practice

1. Determine the solutions for each equation. Identify the solutions on one of the graphs. Then, write the solutions in terms of their respective differences from the axis of symmetry.







a. $8 = x^2 + 3$

b. $7 = x^2$

e. $x^2 + 9 = 13$

c. $2 = x^2 - 1$

d. $x^2 = 11$

f. $14 = x^2 - 1$

Lesson 1 Assignment

2. Estimate the value of each radical expression. Then, rewrite each radical by extracting all perfect squares, when possible.

a. $\sqrt{21}$

b. $\sqrt{80}$

c. $\sqrt{63}$

d. $\sqrt{32}$

e. $\sqrt{98}$

f. $\sqrt{192}$

3. Rewrite each quadratic function as two linear factors using what you know about the difference of two squares.

a. $f(x) = 9x^2 - 16$

b. $f(x) = x^2 - 8$

c. $f(x) = 36x^2 - 1$

d. $f(x) = 25x^2 - 12$

Prepare

Describe the transformations to the graph of the parent function $f(x) = x^2$ given each equation.

1. $y = (x - 4)^2$

2. $y = \frac{1}{2}(x + 1)^2$

3. $y = -(10 + x)^2 - 3$

4. $y = (8 + x)^2 + 1$

2 Solutions to Quadratic Equations in Vertex Form

OBJECTIVES

- Identify solutions to and roots of quadratic equations given in the form $f(x) = (x - c)^2$.
- Identify solutions to and roots of quadratic equations given in the form $f(x) = a(x - c)^2$.
- Identify solutions to and roots of quadratic equations given in the form $f(x) = a(x - c)^2 + d$.
- Identify zeros of quadratic functions written in vertex form.

NEW KEY TERMS

- horizontal asymptote
- extracting square roots

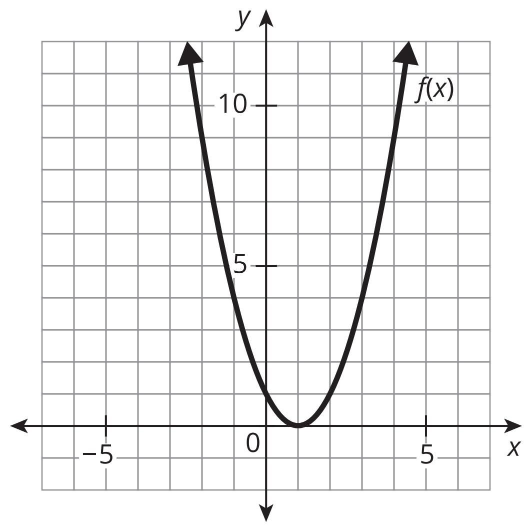
You have explored transformations of quadratic functions and vertex form.
How can you use vertex form and transformations to determine solutions to quadratic equations?

Getting Started

Slide, Slide, Slippity Slide

The coordinate plane shows the graph of the function $f(x) = (x - 1)^2$.

1. Describe the transformation applied to the parent function $f(x) = x^2$ that produces the graph of this function.



Daniela and Amir determined the zeros of the function $f(x) = (x - 1)^2$ algebraically in different ways.

Daniela

$$0 = (x - 1)^2$$

$$0 = (x - 1)(x - 1)$$

The zero product property says that one or both of the factors is equal to 0.

So, $x = 1$.

The equation has a double root at $x = 1$.

Amir

$$(x - 1)^2 = 0$$

$$\sqrt{(x - 1)^2} = \sqrt{0}$$

$$\pm(x - 1) = 0$$

$+(x - 1) = 0$	$-(x - 1) = 0$
$x = 1$	$-x + 1 = 0$
	$-x = -1$
	$x = 1$

The only unique solution for $y = 0$ is $x = 1$.

2. How can you use Daniela's or Amir's work to write solutions to the function in terms of their respective distances from the axis of symmetry?

ACTIVITY 2.1 Solutions for Horizontal Translations

You have used graphs to solve equations. In this activity, you will use the graph of a quadratic equation to determine its solutions.

WORKED EXAMPLE

Consider the equation $(x - 1)^2 = 9$.

You can use the properties of equality to determine the solutions to an equation in this form.

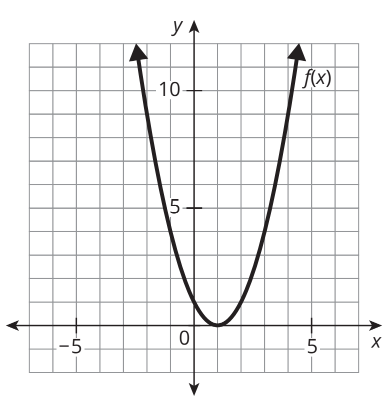
First, take the square root of both sides of the equation and then isolate x.

$$(x - 1)^2 = 9$$
$$\sqrt{(x - 1)^2} = \sqrt{9}$$
$$x - 1 = \pm 3$$
$$x = 1 \pm 3$$

Remember . . .
Solving $(x - 1)^2 = 9$ on a graph means locating where $y = (x - 1)^2$ intersects with $y = 9$.

1. Consider the graph of $y = (x - 1)^2$ in Getting Started.

   a. Graph the equation $y = 9$ on the same graph.

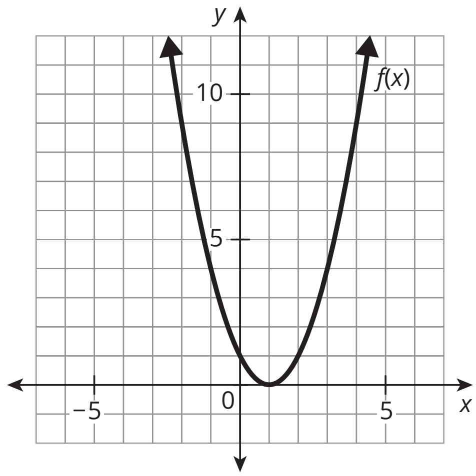


   b. Show the solutions on the graph. Interpret the solutions 1 ± 3 in terms of the axis of symmetry and the points on the parabola $y = (x - 1)^2$.

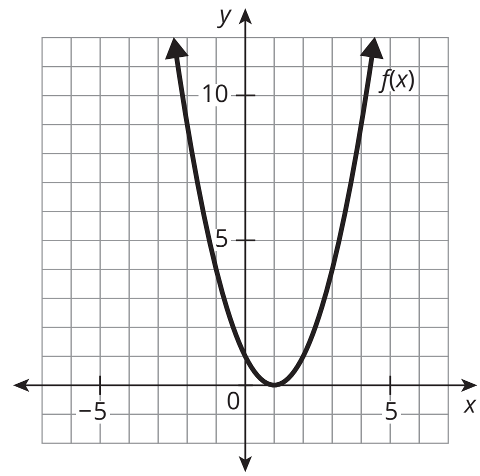
   c. What are the solutions to the equation $(x - 1)^2 = 9$?

2. For each equation, show the solutions on the graph and interpret the solutions in terms of the axis of symmetry and the points on the parabola. Then, write the solutions.

a. $(x - 1)^2 = 4$



b. $(x - 1)^2 = 5$



3. Determine the exact and approximate solutions for each of the given equations.

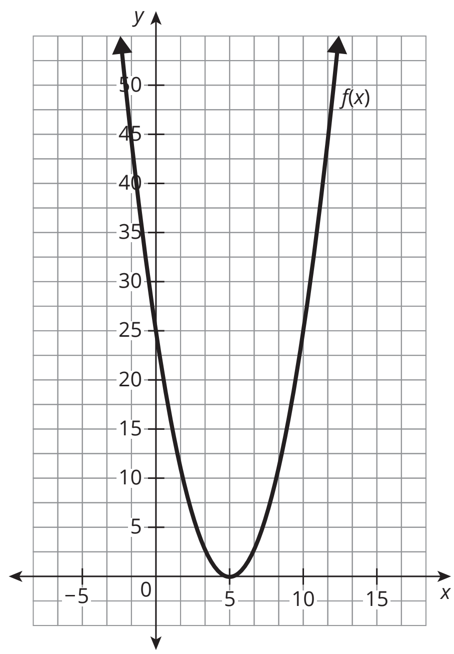
a. $(r + 8)^2 = 83$

b. $(17 - d)^2 = 55$

ACTIVITY
2.2 Solutions for Vertical Dilations

You have seen how to solve an equation for a quadratic function in the form $f(x) = (x - c)^2$, which represents a horizontal translation of the function. In this activity, you will consider quadratic equations with an additional vertical dilation. First, let's start with just a horizontal translation.

1. Consider the function $f(x) = (x - 5)^2$.



   a. Determine the solutions to $0 = (x - 5)^2$. Solve algebraically and label the solution on the graph.

   b. Interpret your solutions in terms of the axis of symmetry and the parabola $y = (x - 5)^2$.

   c. Describe the zeros of this function.

Now let's add a dilation factor.

2. Consider the function $g(x) = 2(x - 5)^2$.

   a. Write $g(x)$ in terms of $f(x)$ and describe the transformation.

   b. Sketch a graph on the same coordinate plane as $f(x)$.

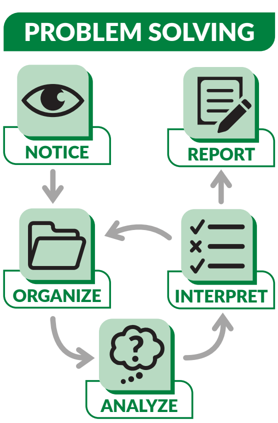
   c. How have the zeros changed from $f(x)$ to $g(x)$?

3. Koda formulated a conjecture about how the solutions of the transformed quadratic equation change from the original equation.

> The solutions of the original function are $x = 5 \pm \sqrt{4}$, so the solutions to the transformed equation will be $x = 5 \pm 2\sqrt{4}$.

Is Koda correct? If so, explain why. If not, describe the correct solutions for the transformed quadratic equation.

4. Make a conjecture. How does changing the *a*-value affect the solutions to the quadratic equations in this form?



5. Solve each quadratic equation. Give both exact and approximate solutions.

a. $(x - 4)^2 = 2$

b. $2(x - 1)^2 = 18$

c. $-2(x - 1)^2 = -18$

d. $4(x + 5)^2 = 21$

e. $-\frac{1}{2}(x + 8)^2 = -32$

f. $\frac{2}{3}(12 - x)^2 = 1$

ACTIVITY 2.3

Solutions for Vertical Translations

You have determined solutions to quadratic equations given an equation in the form $f(x) = a(x - c)^2$. How can you solve a quadratic equation that also includes a vertical translation in the form $f(x) = a(x - c)^2 + d$?

Remember . . .

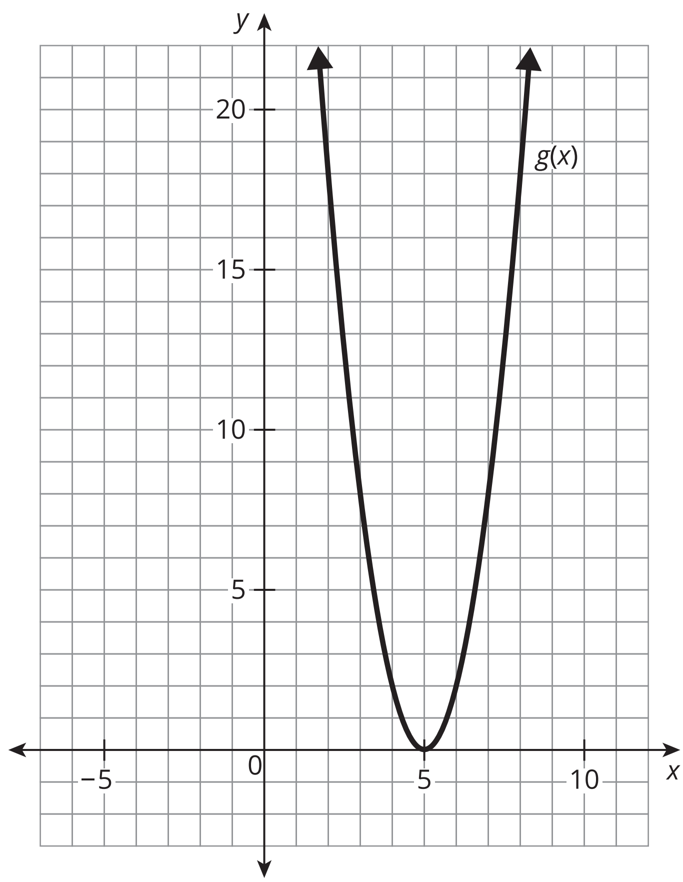
A quadratic function in vertex form is written $f(x) = a(x - k)^2 + h$.

The graph of $g(x) = 2(x - 5)^2$ is shown. You know that the solution to the equation $0 = 2(x - 5)^2$ is $x = 5$.

1. Consider the function $h(x) = 2(x - 5)^2 - 1$.

   a. Write $h(x)$ in terms of $g(x)$ and describe the transformation.

   b. Sketch a graph of $h(x)$ on the same coordinate plane as $g(x)$.



2. Consider the equation $0 = 2(x - 5)^2 - 1$.

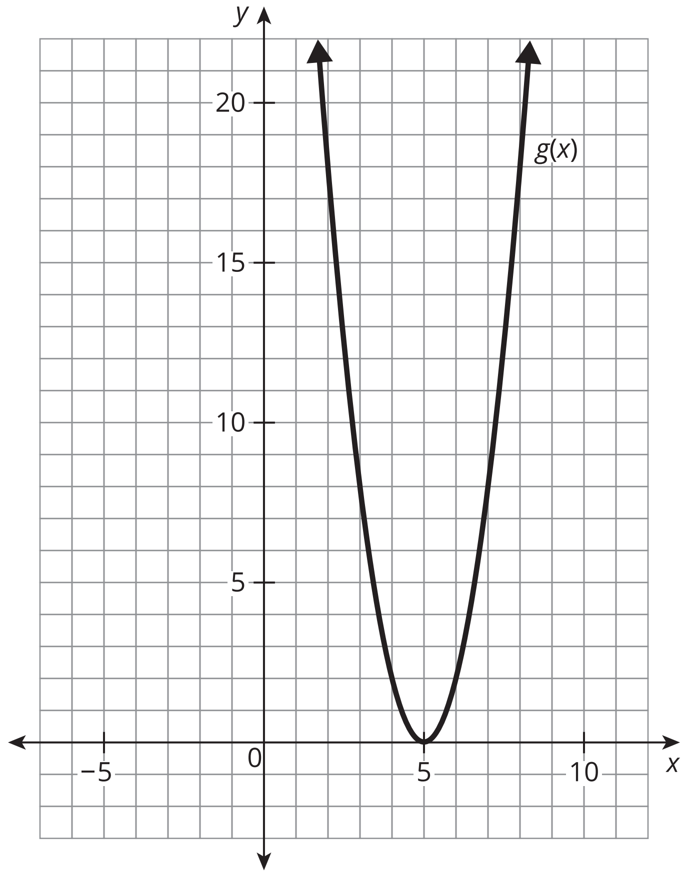
   a. Determine the solution algebraically and label the solution on the graph.

   b. Interpret the solutions in terms of the axis of symmetry and the parabola $y = 2(x - 5)^2 - 1$.

   c. Describe the zeros of this function.

Review the definition of **describe** in the Academic Glossary.

Now, let's investigate the effect of an equation in the form $f(x) = a(x - c)^2 + d$, where $a > 0$ and $d > 0$. Consider the function $j(x) = 2(x - 5)^2 + 1$ graphed, as shown.



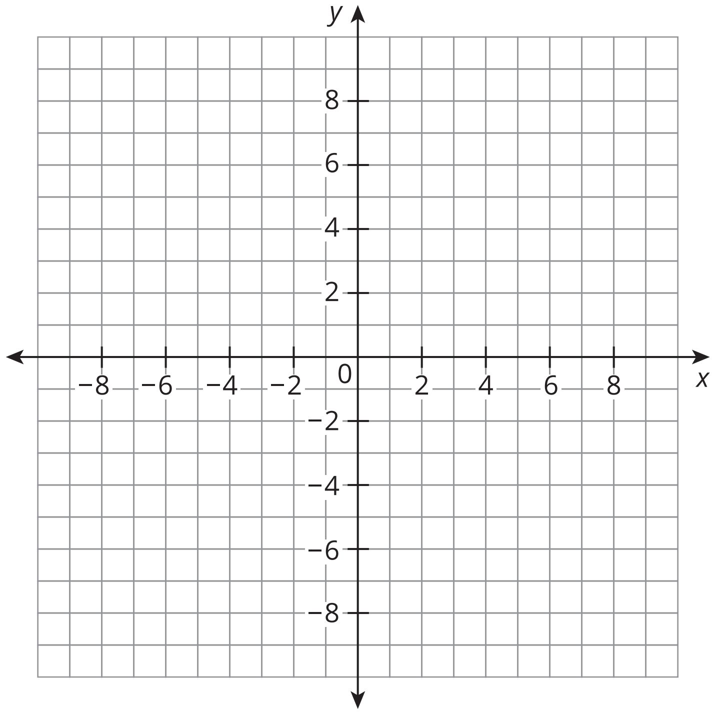
Notice the graph of $j(x)$ does not cross the x-axis, which means there are no real zeros for this function.

3. Solve $0 = 2(x - 5)^2 + 1$ algebraically to show that x is not a real number.

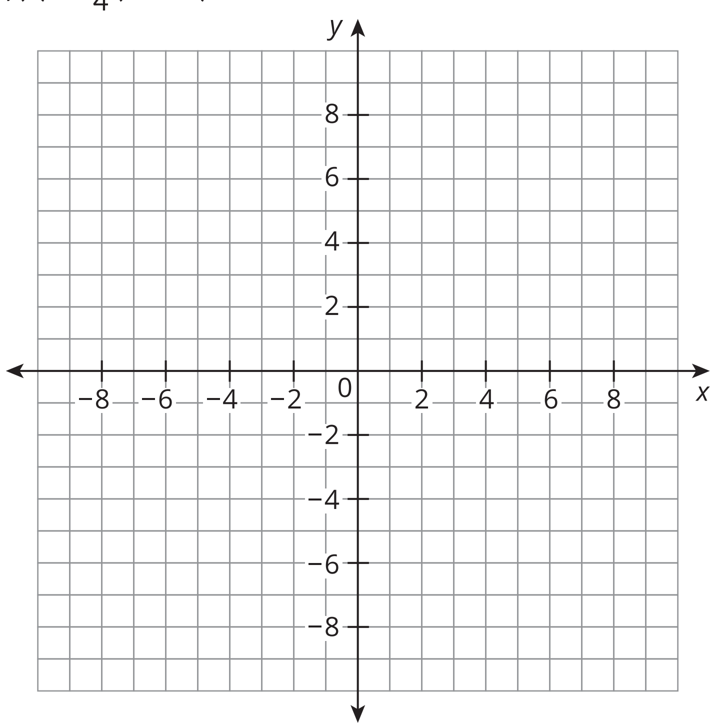
4. Sketch a graph of each quadratic function. Determine the types of zeros of each function. Solve algebraically and interpret on the graph in terms of the axis of symmetry and the points on the parabola.

A quadratic function can have 1 unique real zero, 2 real zeros, or no real zeros.

a. $f(x) = -3(x - 2)^2 + 4$



b. $f(x) = \frac{1}{4}(x + 5)^2 + 2$



Ask Yourself . . .
What is the relationship of the zeros of the function and the *x*-intercepts of the graph?

Talk the Talk

Spell It Out

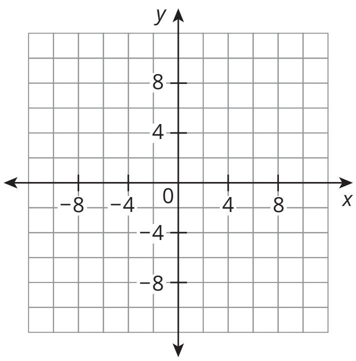
1. Describe the solution of any quadratic equation in the form $(x - c)^2 = 0$.

2. Describe the solution of any quadratic equation in the form $(x - c)^2 + d = 0$.

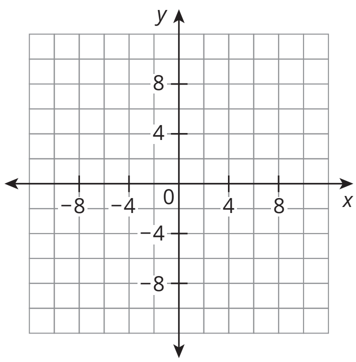
3. Describe the solution of any quadratic equation in the form $a(x - c)^2 + d = 0$.

4. Write an equation and sketch a graph that shows each number of zeros.

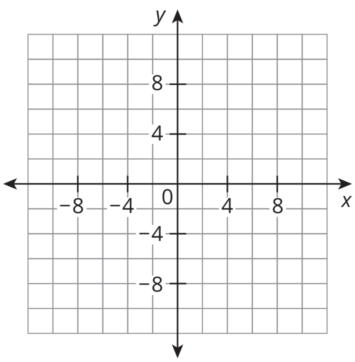
a. 1 unique real zero



b. 2 real zeros



c. no real zeros



Lesson 2 Assignment

Write

Describe the number of possible real zeros for any quadratic function.

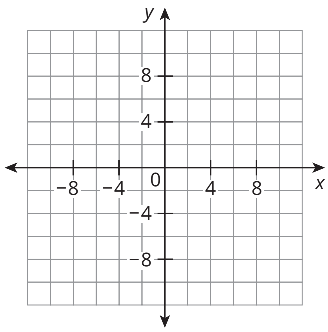
Remember

The solutions to a quadratic equation can be represented as the axis of symmetry plus or minus its distance to the parabola.

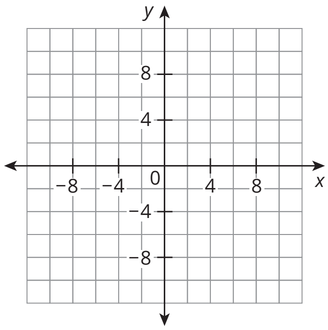
Practice

1. Sketch a graph of each quadratic function. Determine the zeros of each function and write each in terms of the axis of symmetry and its distance from the parabola.

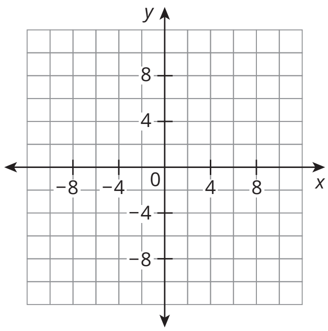
a. $f(x) = (x - 3)^2$



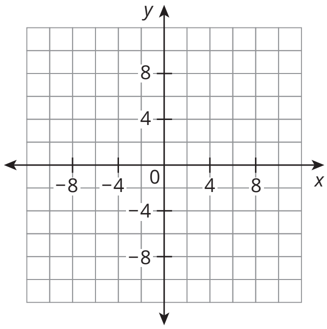
b. $f(x) = (x + 5)^2$



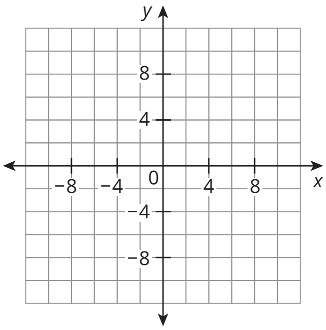
c. $f(x) = \left(x - \frac{1}{2}\right)^2$



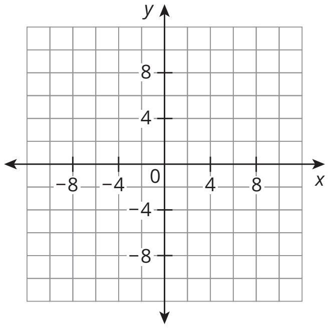
d. $f(x) = (x - 6)^2$



e. $f(x) = \left(x + \frac{15}{7}\right)^2$



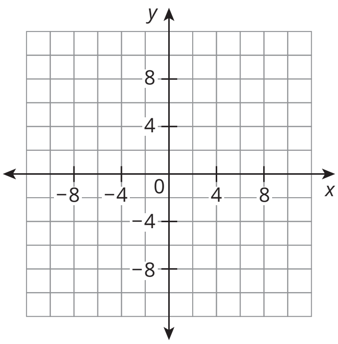
f. $f(x) = (x + 7)^2$



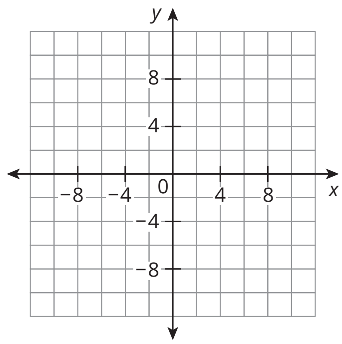
Lesson 2 Assignment

2. Sketch a graph of each quadratic function. Determine the zeros of each function and write each in terms of the axis of symmetry and its distance from the parabola.

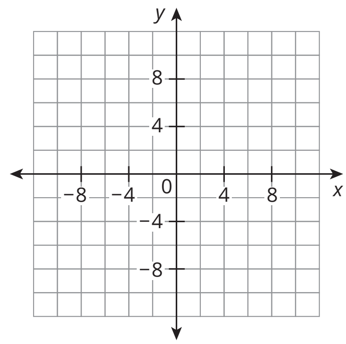
a. $f(x) = 2(x - 1)^2 - 1$



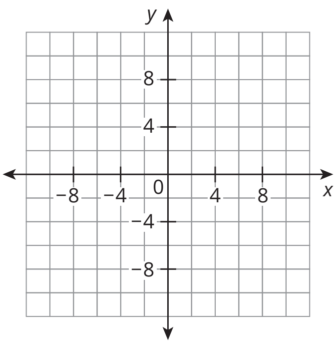
b. $f(x) = \frac{1}{2}(x + 2)^2 - 5$



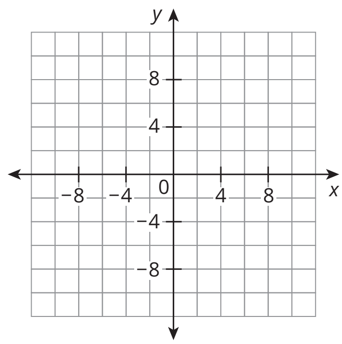
c. $f(x) = 4\left(x + \frac{1}{3}\right)^2 - 1$



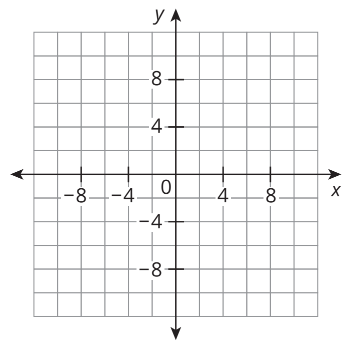
d. $f(x) = -3(x - 6)^2$



e. $f(x) = \frac{3}{4}(x + 5)^2 - \frac{2}{3}$



f. $f(x) = (x - 4)^2 - 2$



Prepare

Use the distributive property to determine each product.

1. $(x + 1)(x + 2)$

2. $(x + 4)(x - 5)$

3. $(2x - 3)(x - 4)$

4. $(x + 2)^2$

3 Factoring and Completing the Square

OBJECTIVES

- Factor out the greatest common factor (GCF) of polynomials.
- Rewrite quadratic equations of the form $x^2 + bx$ in vertex form by completing the square.
- Factor quadratic trinomials to determine the roots of quadratic equations and to rewrite quadratic functions in forms that reveal different key characteristics.
- Demonstrate the reasoning behind the method of completing the square and use the method to determine the roots of quadratic equations of the form $ax^2 + bx + c$.

NEW KEY TERM

- completing the square

You have solved many different quadratic equations written as binomials.

How can you solve trinomial quadratic equations?

Getting Started

LOL the GCF Again

In previous lessons, you multiplied two linear expressions to determine a quadratic expression. You have also rewritten quadratics in factored form.

You may remember that one way to factor an expression is to factor out the greatest common factor.

WORKED EXAMPLE

Consider the polynomial $3x + 15$. You can factor out the greatest common factor of the two terms, 3.

$$3x + 15 = 3x + 3(5)$$

$$= 3(x + 5)$$

$$3x + 15 = 3(x + 5)$$

1. Factor out the greatest common factor for each polynomial, when possible.

   a. $4x + 12$

   b. $x^2 - 5x$

   c. $3x^2 - 9x - 3$

   d. $-x - 7$

   e. $2x - 11$

   f. $5x^2 - 10x + 5$

ACTIVITY 3.1

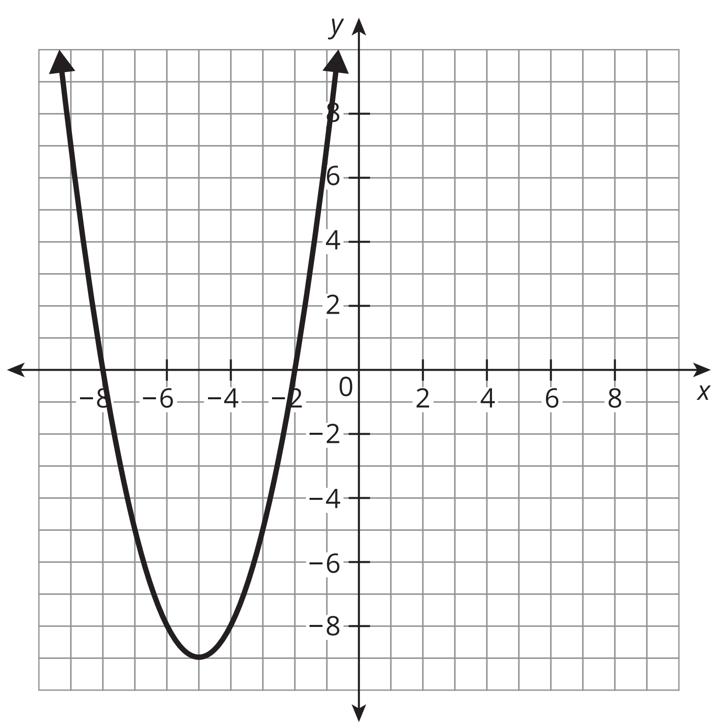
Factoring Trinomials

You have used special products—the difference of two squares and perfect square trinomials—to rewrite trinomials in factored form. In this activity, you will rewrite trinomials that are not special products in factored form.

1. Consider the equation $y = x^2 + 10x + 16$.

   a. Use the graph to identify the roots of the equation.

   b. Rewrite the original equation in factored form.



Let's consider a strategy to factor a trinomial without graphing.

You can use a multiplication table to factor trinomials.

WORKED EXAMPLE

Factor the trinomial $x^2 + 10x + 16$.
Start by writing the leading term (x^2) and the constant term (16) in the table.

·		
	x^2	
		16

Determine the two factors of the leading term and write them in the table.

·	x	
x	x^2	
		16

Determine the factor pairs of the constant term. The factors of 16 are (1)(16), (2)(8), and (4)(4). Experiment with factors of the constant term to determine the pair whose sum is the coefficient of the middle term, 10.

·	x	8
x	x^2	$8x$
2	$2x$	16

The sum of $2x$ and $8x$ is $10x$.
So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

2. Explain why the other factor pairs for $c = 16$ do not work.

3. Use the Worked Example to factor each trinomial.

a. $x^2 + 17x + 16$

·		
	x^2	
		16

b. $x^2 + 6x - 16$

·		
	x^2	
		-16

c. $x^2 - 6x - 16$

·		
	x^2	
		-16

4. Factor each trinomial.

a. $x^2 + 5x - 24$

b. $x^2 - 3x - 28$

5. Consider the two examples shown.

Destiny

$2x^2 - 3x - 5$

·	x	1
$2x$	$2x^2$	$2x$
-5	$-5x$	-5

$2x^2 - 3x - 5 = (2x - 5)(x + 1)$

Gabriela

$2x^2 + 3x - 5$

·	x	-1
$2x$	$2x^2$	$-2x$
5	$5x$	-5

$2x^2 + 3x - 5 = (2x + 5)(x - 1)$

a. Compare the two given trinomials. What is the same and what is different about the values of *a*, *b*, and *c*?

Remember . . .
The standard form of a quadratic equation is a trinomial in the form $y = ax^2 + bx + c$.

b. Compare the factored form of each trinomial. What do you notice?

6. Choose from the list to write the correct factored form for each trinomial.

a. $x^2 + 5x + 4 =$ ________________
$x^2 - 5x + 4 =$ ________________
$x^2 + 3x - 4 =$ ________________
$x^2 - 3x - 4 =$ ________________

- $(x + 1)(x - 4)$
- $(x + 1)(x + 4)$
- $(x - 1)(x + 4)$
- $(x - 1)(x - 4)$

b. $2x^2 + 7x + 3 =$ ________________
$2x^2 - 7x + 3 =$ ________________
$2x^2 + 5x - 3 =$ ________________
$2x^2 - 5x - 3 =$ ________________

- $(2x - 1)(x - 3)$
- $(2x - 1)(x + 3)$
- $(2x + 1)(x + 3)$
- $(2x + 1)(x - 3)$

c. $x^2 + 7x + 10 =$ ________________
$x^2 - 7x + 10 =$ ________________
$x^2 + 3x - 10 =$ ________________
$x^2 - 3x - 10 =$ ________________

- $(x - 2)(x + 5)$
- $(x + 2)(x + 5)$
- $(x - 2)(x - 5)$
- $(x + 2)(x - 5)$

7. Analyze the signs of each quadratic expression written in standard form and the operations in the binomial factors in Question 6. Then, complete each sentence with a phrase from the box.

the same	different	both positive	both negative
	one positive and one negative		

a. If the constant term is positive, then the operations in the binomial factors are ________________.

b. If the constant term is positive and the middle term is positive, then the operations in the binomial factors are ________________.

c. If the constant term is positive and the middle term is negative, then the operations in the binomial factors are ________________.

d. If the constant term is negative, then the operations in the binomial factors are ________________.

e. If the constant term is negative and the middle term is positive, then the operations in the binomial factors are

________________________________.

f. If the constant term is negative and the middle term is negative, then the operations in the binomial factors are

________________________________.

8. Factor each quadratic expression.

a. $x^2 + 8x + 15 =$ ________________

$x^2 - 8x + 15 =$ ________________

$x^2 + 2x - 15 =$ ________________

$x^2 - 2x - 15 =$ ________________

b. $x^2 + 10x + 24 =$ ________________

$x^2 - 10x + 24 =$ ________________

$x^2 + 2x - 24 =$ ________________

$x^2 - 2x - 24 =$ ________________

9. Hannah, Madison, and Ana were asked to factor the trinomial $15 + 2x - x^2$.

Hannah

$15 + 2x - x^2$

$(5 - x)(3 + x)$

Madison

$15 + 2x - x^2$

$(5 - x)(3 + x)$

$(x - 5)(x + 3)$

Ana

$15 + 2x - x^2$

$-x^2 + 2x + 15$

$-(x^2 - 2x - 15)$

$-(x - 5)(x + 3)$

Who's correct? Explain how the correct student(s) determined the factors. For the student(s) who is/are not correct, state why and make the correction.

Maria and Luis were working together to factor the trinomial $4x^2 + 22x + 24$. They first noticed that there was a greatest common factor and rewrote the trinomial as

$$2(2x^2 + 11x + 12).$$

Next, they considered the factor pairs for $2x^2$ and the factor pairs for 12.

$2x^2$: $(2x)(x)$
12: $(1)(12)$
$(2)(6)$
$(3)(4)$

Maria listed out all the possible combinations.

$2(2x + 1)(x + 12)$
$2(2x + 12)(x + 1)$

$2(2x + 2)(x + 6)$
$2(2x + 6)(x + 2)$

$2(2x + 3)(x + 4)$
$2(2x + 4)(x + 3)$

Luis immediately eliminated four out of the six possible combinations because the terms of one of the linear expressions contained common factors.

$2(2x + 1)(x + 12)$
~~$2(2x + 12)(x + 1)$~~

~~$2(2x + 2)(x + 6)$~~
~~$2(2x + 6)(x + 2)$~~

$2(2x + 3)(x + 4)$
~~$2(2x + 4)(x + 3)$~~

10. Explain Luis's reasoning. Then, circle the correct factored form of $4x^2 - 22x + 24$.

ACTIVITY 3.2

Solving Quadratic Equations by Factoring

You have used properties of equality to solve equations in the forms shown.

$$y = x^2 + d$$
$$y = (x - c)^2$$
$$y = a(x - c)^2$$
$$y = a(x - c)^2 + d$$

Let's consider strategies to solve quadratics in the form $y = ax^2 + bx + c$ using the factoring strategies you just learned.

WORKED EXAMPLE

You can calculate the roots for the quadratic equation $x^2 - 4x = -3$.

$$x^2 - 4x = -3$$
$$x^2 - 4x + 3 = -3 + 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

$(x - 3) = 0$	and	$(x - 1) = 0$
$x - 3 + 3 = 0 + 3$	and	$x - 1 + 1 = 0 + 1$
$x = 3$	and	$x = 1$

Think about . . .
What is the connection between the Worked Example and determining the roots from factored form, $y = a(x - r_1)(x - r_2)$?

1. Consider the Worked Example. Why is 3 added to both sides in the first step?

Remember . . .
The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

2. Determine each student's error, then solve each equation correctly.

Abby

$x^2 + 6x = 7$

$x(x + 6) = 7$

$x = 7$ and $x + 6 = 7$

$x = 1$

William

$x^2 + 5x + 6 = 6$

$(x + 2)(x + 3) = 6$

$x + 2 = 6$ and $x + 3 = 6$

$x = 4$ and $x = 3$

3. Use factoring to solve each quadratic equation, when possible.

a. $x^2 - 8x + 12 = 0$

b. $x^2 - 5x - 24 = 0$

c. $x^2 + 10x - 75 = 0$

d. $x^2 - 11x = 0$

e. $x^2 + 8x = -7$

f. $x^2 - 5x = 13x - 81$

g. $\frac{2}{3}x^2 - \frac{5}{6}x = 0$

h. $f(x) = x^2 + 10x + 12$

Think about . . .

What efficiency strategies did you use to solve linear equations with fractional coefficients?

4. Describe the different strategies and reasoning that Angel and Huyen used to solve $4x^2 - 25 = 0$.

Angel

$4x^2 - 25 = 0$

$4x^2 = 25$

$x^2 = \frac{25}{4}$

$x = \pm\sqrt{\frac{25}{4}}$

$x = \pm\frac{5}{2}$

Huyen

$4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$

$2x - 5 = 0$ and $2x + 5 = 0$

$2x = 5 \qquad 2x = -5$

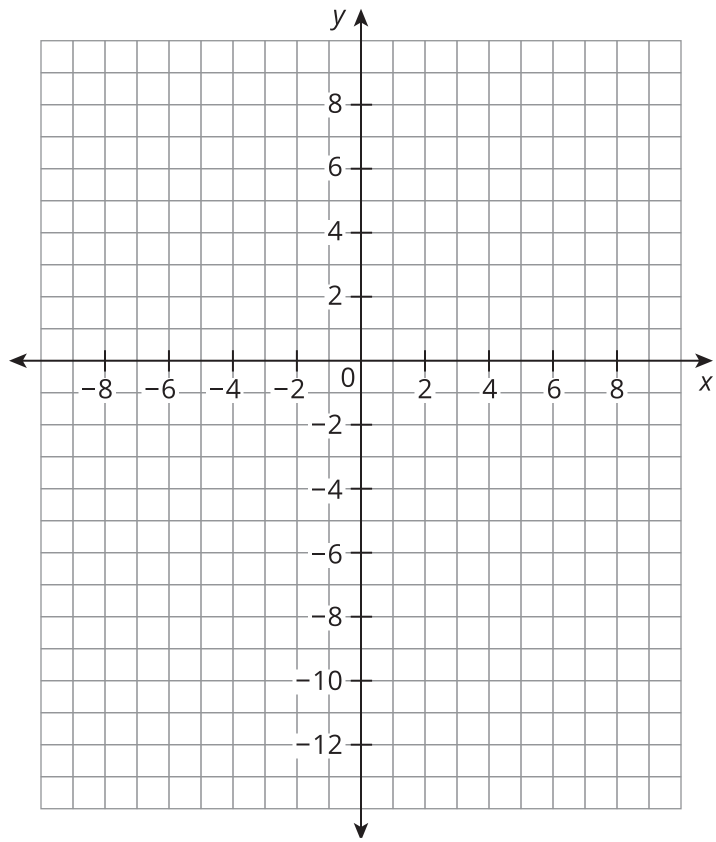
$x = \frac{5}{2}$ and $x = -\frac{5}{2}$

ACTIVITY
3.3

Completing the Square

When you cannot factor a quadratic function, does that mean it does not have zeros?

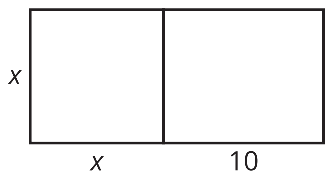
1. Consider the quadratic equation $y = x^2 + 10x + 12$. Use technology to graph the equation and then sketch it on the coordinate plane. Does this function have zeros? Explain your reasoning.



The quadratic function you graphed has zeros but cannot be factored, so you must consider another method for calculating its zeros. You can use your understanding of the relationship among the coefficients of a perfect square trinomial to construct a procedure to solve any quadratic equation.

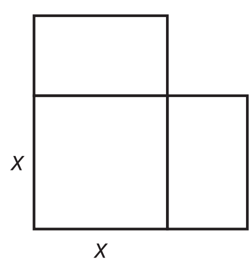
Previously, you factored trinomials of the form $a^2 + 2ab + b^2$ as the perfect square $(a + b)^2$. This knowledge can help you develop a procedure to solve any quadratic equation.

2. The expression $x^2 + 10x$ can be represented geometrically, as shown. Write the area of each rectangle within the diagram.



3. This figure can now be modified into the shape of a square by splitting the second rectangle in half and rearranging the pieces.

   a. Complete the side length labels for the split rectangle and write the area of each piece within the diagram.

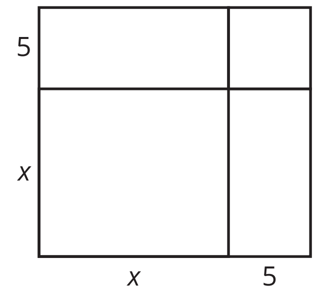


Ask Yourself . . .
Why do you divide the second rectangle in half?

   b. Do the two figures represent the same expression? Explain your reasoning.

Review the definition of **represent** in the Academic Glossary.

c. Complete the figure to form a square. Label the area of the piece you added.



d. Add the term *representing* the additional area to the original expression. What is the new expression?

e. Factor the new expression.

The process you just worked through is a method known as *completing the square*. **Completing the square** is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

4. Draw a model to complete the square for each expression. Then, factor the resulting trinomial.

a. $x^2 + 8x$

b. $x^2 + 5x$

5. Analyze your work in Question 4.

   a. Explain how to complete the square on an expression of the form $x^2 + bx$, where b is an integer.

   b. Describe how the coefficient of the middle term, b, is related to the constant term, c, in each trinomial you wrote in Question 4.

6. Use the descriptions you provided in Question 5 to determine the unknown second or third term to make each expression a perfect square trinomial. Then, write the expression as a binomial squared.

   a. $x^2 - 8x +$ ________________ $=$ ________________

   b. $x^2 + 5x +$ ________________ $=$ ________________

   c. $x^2 -$ ________________ $+ 100 =$ ________________

   d. $x^2 +$ ________________ $+ 144 =$ ________________

ACTIVITY 3.4

Completing the Square to Determine Roots

So far, you have considered quadratic equations that can be rewritten by completing the square or factoring a trinomial.

You can use the completing the square method to determine the roots of a quadratic equation that cannot be factored.

WORKED EXAMPLE

Determine the roots of the equation $x^2 + 10x + 12 = 0$.

Isolate $x^2 + 10x$. You can complete the square and rewrite this as a perfect square trinomial.

$$x^2 + 10x + 12 - 12 = 0 - 12$$
$$x^2 + 10x = -12$$

Determine the constant term that would complete the square. Add this term to both sides of the equation.

$$x^2 + 10x + \underline{\quad} = -12 + \underline{\quad}$$
$$x^2 + 10x + 25 = -12 + 25$$
$$x^2 + 10x + 25 = 13$$

Factor the left side of the equation.

$$(x + 5)^2 = 13$$

Determine the square root of each side of the equation.

$$\sqrt{(x + 5)^2} = \pm\sqrt{13}$$
$$x + 5 = \pm\sqrt{13}$$

Set the factor of the perfect square trinomial equal to each square root of the constant. Solve for x.

$x + 5 = \sqrt{13}$ and $x + 5 = -\sqrt{13}$

$x = -5 + \sqrt{13}$ and $x = -5 - \sqrt{13}$

$x \approx -1.39$ and $x \approx -8.61$

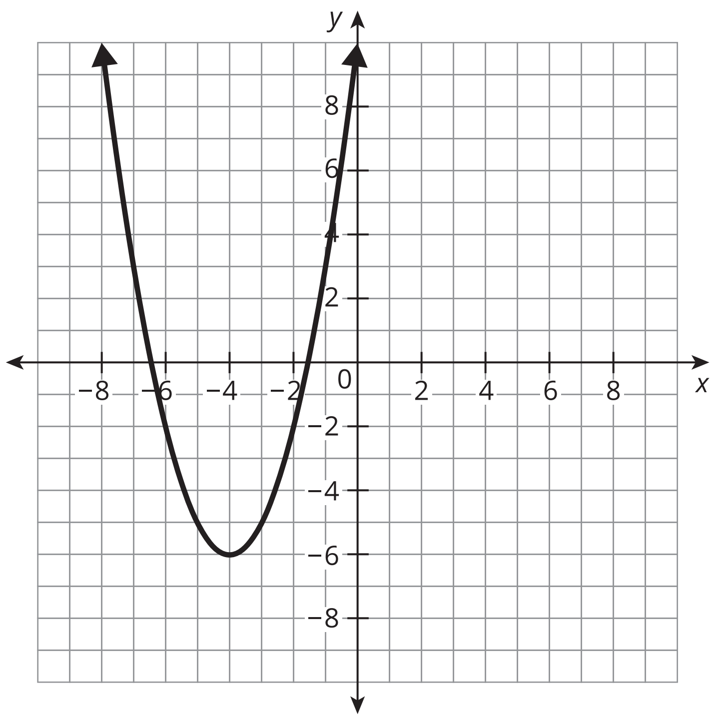
The roots are approximately (−1.39) and (−8.61).

Ask Yourself . . .
How was equality of the equation maintained through the completing the square process?

1. Consider the equation $y = x^2 + 8x + 10$.

   a. Use this method to determine the roots of the equation.
   Show your work.

   b. Use your work to label the zeros on the graph of the function
   $f(x) = x^2 + 8x + 10$.



2. Determine the roots of each equation by completing the square.

   a. $x^2 - 6x + 4 = 0$

   b. $x^2 - 12x + 6 = 0$

ACTIVITY 3.5

Rewriting a Quadratic in Vertex Form

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

WORKED EXAMPLE

Consider the equation $y = ax^2 + bx + c$.

Step 1: $y - c = ax^2 + bx$

Step 2: $y - c = a\left(x^2 + \frac{b}{a}x\right)$

Step 3: $y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$

Step 4: $y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Step 5: $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

Notice that the *a*-value was factored out before completing the square!

1. Explain why $a\left(\frac{b}{2a}\right)^2$ was added to the left side of the equation in Step 3.

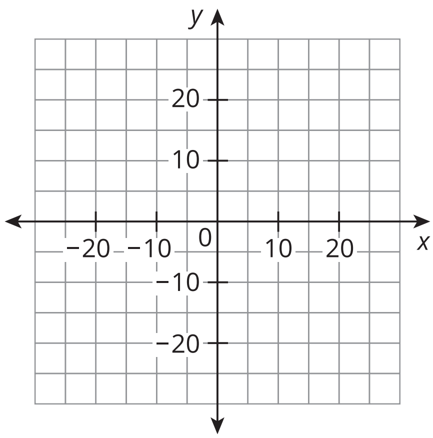
2. Given a quadratic function in the form $y = ax^2 + bx + c$:

   a. Identify the axis of symmetry.

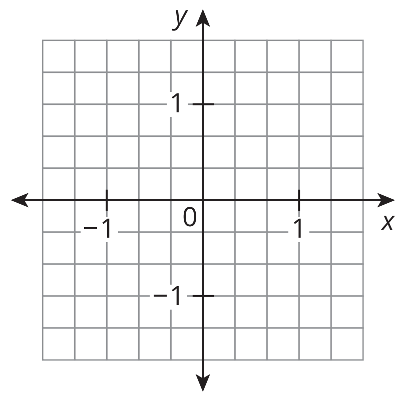
   b. Identify the location of the vertex.

3. Rewrite each quadratic equation in vertex form. Then, identify the zeros and sketch a graph of each function. Write the zeros in terms of the axis of symmetry and the parabola.

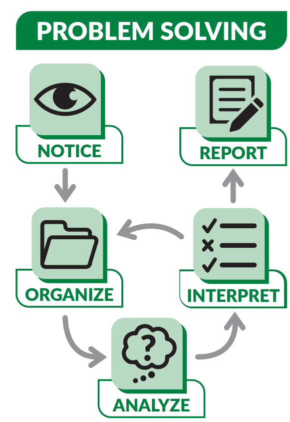
a. $y = x^2 + 8x - 9$



b. $y = 3x^2 + 2x - 1$



4. A ball is thrown straight up from 4 feet above the ground with a velocity of 32 feet per second. The height of the ball over time can be modeled with the function $h(t) = -16t^2 + 32t + 4$. What is the maximum height of the ball?



5. Malik is fencing in a rectangular plot outside of his back door so that he can let his dogs out to play. He has 60 feet of fencing and only needs to place it on three sides of the rectangular plot because the fourth side will be bound by his house. What dimensions should Malik use for the plot so that the maximum area is enclosed? What is the maximum area? Draw a diagram to support your work.

Ask Yourself . . .
Did you complete all the steps in the problem-solving model?

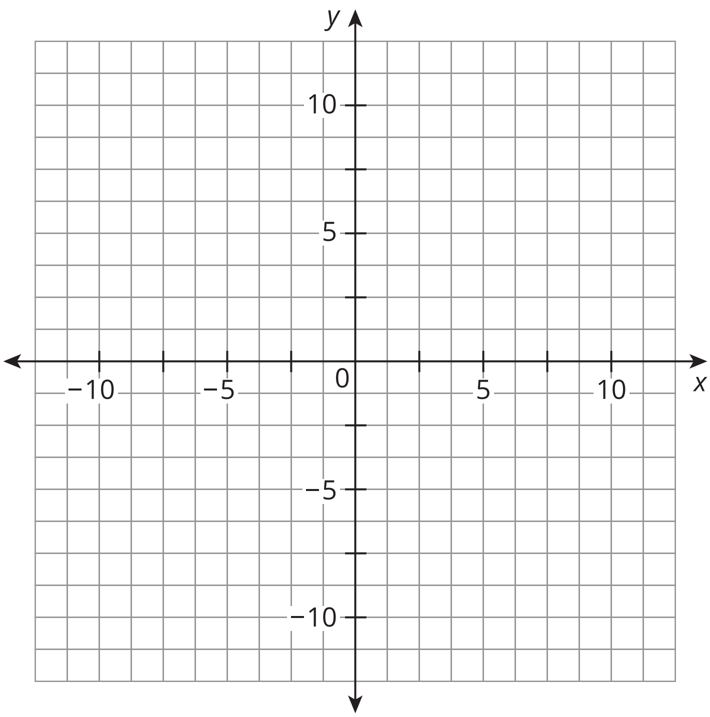
Talk the Talk

Play It Again

1. Consider the quadratic equation $y = x^2 - 4x - 5$.

   a. Rewrite the equation in factored form and vertex form.

   b. Graph the function. Identify the vertex, x- and y-intercepts, and the axis of symmetry. Then, explain how these are evident in each form of the equation.



Lesson 3 Assignment

Write

Describe the process to solve a quadratic equation by factoring.

Remember

- Completing the square is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.
- Given a quadratic equation in the form $y = ax^2 + bx + c$, the vertex of the function is located at $x = \frac{-b}{2a}$ and $y = c - \frac{b^2}{4a}$.

Practice

1. Solve each equation.

   a. $0 = x^2 - 7x - 18$

   b. $x^2 + 10x = 39$

   c. $0 = x^2 - 10x + 12$

   d. $2x^2 + 4x = 0$

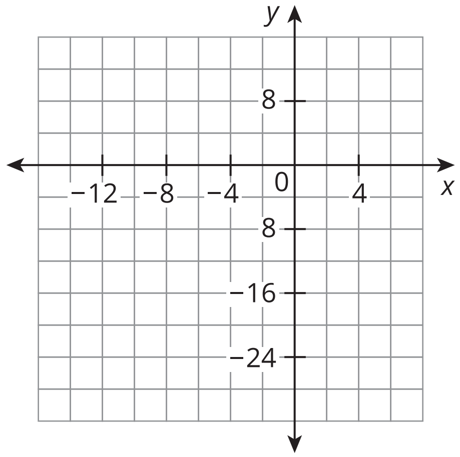
   e. $3x^2 - 22x + 7 = 0$

2. Determine the roots of the equation $y = x^2 + 9x + 3$. Check your solutions.

Lesson 3 Assignment

3. Consider the equation $y = 2x^2 + 10x - 8$.

   a. Graph the equation.



   b. Use your graph to estimate the solutions to the equation. Explain how you determined your answer.

   c. Two students completed the square to determine the solutions to this equation. Their work is shown. Who is correct? Explain your reasoning.

Lesson 3 Assignment

Student 1

$$y = 2x^2 + 10x - 8$$

$$2x^2 + 10x - 8 = 0$$

$$2x^2 + 10x = 8$$

$$2x^2 + 10x + 25 = 8 + 25$$

$$(2x + 5)^2 = 33$$

$$\sqrt{(2x + 5)^2} = \pm\sqrt{33}$$

$$2x + 5 = \pm\sqrt{33}$$

$$x = \frac{-5 \pm \sqrt{33}}{2}$$

$x < -5.372$ and $x < 0.372$

Student 2

$$y = 2x^2 + 10x - 8$$

$$2x^2 + 10x - 8 = 0$$

$$\frac{2x^2 + 10x - 8}{2} = 0$$

$$x^2 + 5x = 4$$

$$x^2 + 5x + \frac{25}{4} = 4 + \frac{25}{4}$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{41}{4}$$

$$\sqrt{\left(x + \frac{5}{2}\right)^2} = \pm\sqrt{\frac{41}{4}}$$

$$x + \frac{5}{2} = \pm\frac{\sqrt{41}}{2}$$

$$x = \frac{-5 \pm \sqrt{41}}{2}$$

$x < -5.702$ and $x < 0.702$

d. Compare the solutions. Identify what the student who got the correct answer did that allowed them to correctly complete the square.

e. Write a statement about the value of the coefficient of the x^2-term before you can complete the square.

4. Determine the roots of the equation $y = 3x^2 + 24x - 6$. Check your solutions.

Lesson 3 Assignment

5. Determine the roots and the location of the vertex of $y = x^2 + 20x + 36$. Write the zeros in terms of the axis of symmetry and the parabola.

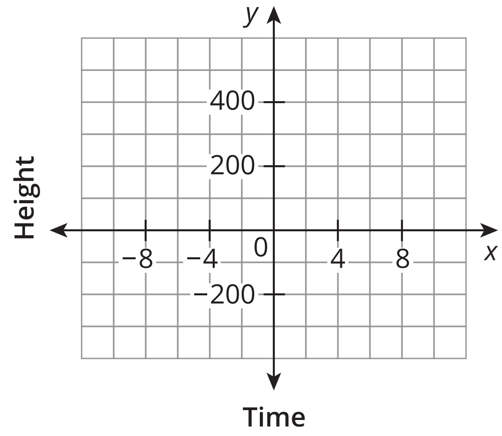
Prepare

A bucket of paint falls from the top of a skyscraper that is 564 feet tall.

1. Write a quadratic function to represent the height of the can over time.

2. Use technology to graph the function.

3. How many seconds will it take for the can of paint to hit the ground?



4 The Quadratic Formula

OBJECTIVES

- Derive the quadratic formula from a quadratic equation written in standard form.
- Connect the quadratic formula to a graphical representation.
- Use the discriminant of the quadratic formula to determine the number of roots or zeros.
- Use the quadratic formula to determine roots and zeros.

NEW KEY TERMS

- quadratic formula
- discriminant

You know several strategies to solve quadratic equations, depending on the structure of the equation.

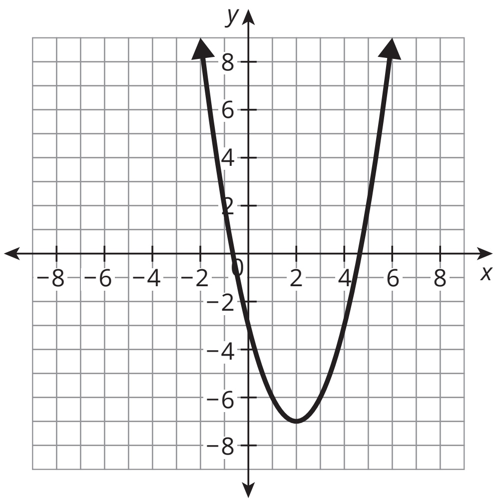
Is there a single strategy that will work to solve any quadratic equation?

Getting Started

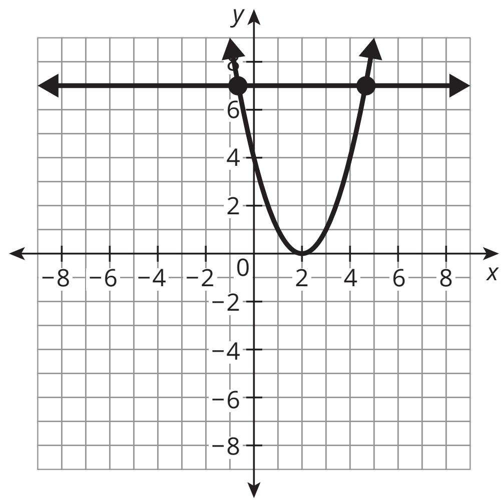
Really, They Aren't the Same

Consider each graph.

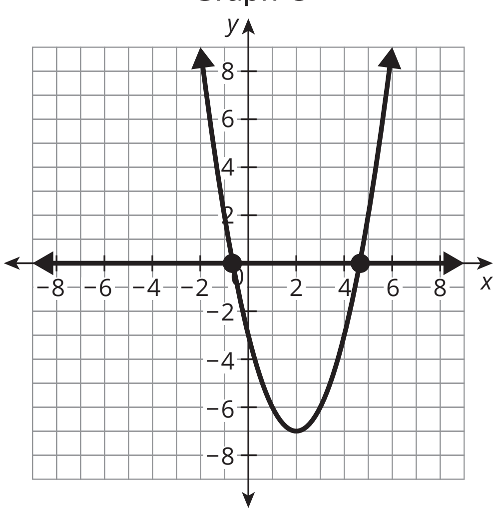
Graph A



Graph B



Graph C



1. Match each equation to its corresponding graph.

   a. $(x - 2)^2 - 7 = 0$

   b. $y = (x - 2)^2 - 7$

   c. $(x - 2)^2 = 7$

2. How do each of the graphs show solutions? How are the solutions related to the axis of symmetry?

ACTIVITY
4.1 Introducing the Quadratic Formula

In the previous lesson, you took the standard form of a quadratic equation, $y = ax^2 + bx + c$, and rewrote it in vertex form, $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$, by completing the square to determine the vertex and axis of symmetry for graphing purposes.

Now, let's take the standard form of a quadratic equation, $y = ax^2 + bx + c$, and set $y = 0$ to determine the roots. You can complete the square to solve for the x-values when $y = 0$.

WORKED EXAMPLE

Write the equation in standard form with $y = 0$.	$ax^2 + bx + c = 0$
Complete the square.	$ax^2 + bx = -c$ $x^2 + \frac{b}{a}x = -\frac{c}{a}$ $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$
Rewrite the right side of the equation.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$
Now that the equation is written in the form $(x - c)^2 = q$, the square root can be taken on each side. Extract the square roots. Solve for x. These are the roots for the quadratic equation in the standard form, $ax^2 + bx + c = 0$.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ $\quad$ $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$

This approach can be taken one step further and rewritten as a single fraction.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Think About . . .

So really, the quadratic formula is just taking the standard form of a quadratic equation and isolating, or solving for, x.

This equation is known as the *quadratic formula*. The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

You can use the quadratic formula to determine the zeros of the function $f(x) = -4x^2 - 40x - 99$.

WORKED EXAMPLE

Rewrite the function as an equation to be solved for x when $y = 0$.	$-4x^2 - 40x - 99 = 0$
Determine the values of a, b, and c.	$a = -4$, $b = -40$, $c = -99$
Substitute the values into the quadratic formula.	$x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(-4)(-99)}}{2(-4)}$
Perform operations to rewrite the expression.	$x = \frac{40 \pm \sqrt{1600 - 1584}}{-8}$ $x = \frac{40 \pm \sqrt{16}}{-8}$ $x = \frac{40 \pm 4}{-8}$ $x = \frac{40 + 4}{-8}$ and $x = \frac{40 - 4}{-8}$ $x = \frac{44}{-8}$ and $x = \frac{36}{-8}$ $x = -5.5$ and $x = -4.5$
Interpret the solution.	The zeros of the function $f(x) = -4x^2 - 40x - 99$ are $x = -5.5$ and $x = -4.5$.

The baseball team has a new promotional activity to encourage fans to attend games: launching free T-shirts! They can launch a T-shirt in the air with an initial velocity of 91 feet per second from $5\frac{1}{2}$ feet off the ground (the height of the team mascot).

A T-shirt's height can be modeled with the quadratic function $h(t) = -16t^2 + 91t + 5.5$, where t is the time in seconds and $h(t)$ is the height of the launched T-shirt in feet. They want to know how long it will take for a T-shirt to land back on the ground after being launched (if no fans grab it before then!).

Ask Yourself . . .
What would a sketch showing the height of the T-shirt over time look like?

1. Why does it make sense to use the quadratic formula to solve this problem?

2. Use the quadratic formula to determine how long it will take for a T-shirt to land back on the ground after being launched.

Ask Yourself . . .
Do you think an exact solution or an approximate solution is more appropriate for this context?

3. Classify your solutions as rational or irrational.

ACTIVITY 4.2

Interpreting the Quadratic Formula Graphically

You used the quadratic formula to solve a quadratic equation. Let's connect the quadratic formula to the graph. Remember, the quadratic formula can be written to show two roots.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

How do these roots, $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ relate to the graph?

1. What does the first term of each root represent on the graph?

2. The second term of each root represents the distance the root lies from the axis of symmetry. Why is the second term in each root the same except for the sign?

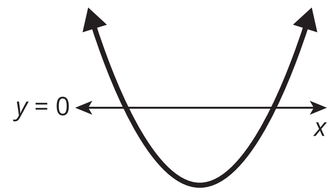
Let's analyze how the structure of the quadratic formula is evident in the graphical representation of the zeros of a quadratic function.

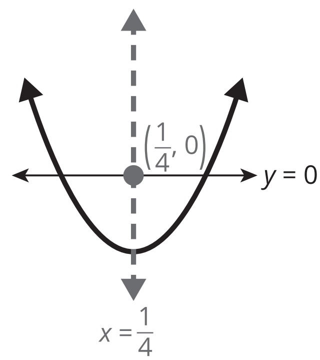
WORKED EXAMPLE

Consider this graphical representation to determine the real roots of the quadratic equation $y = 2x^2 - x - 15$.

<table>
<tr><th>Steps</th><th>Graph</th></tr>
<tr><td>Set y equal to zero and identify the values of a, b, and c.

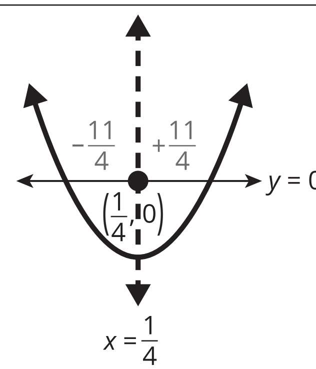
$0 = 2x^2 - x - 15$

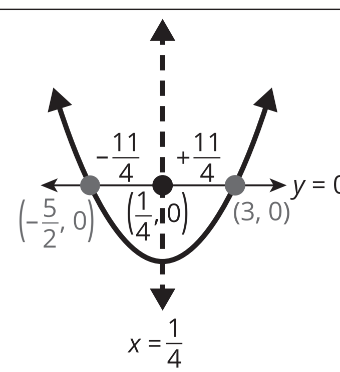
$a = 2 \quad b = -1 \quad c = -15; \quad a > 0$</td><td>
</td></tr>
<tr><td>Identify the axis of symmetry and label the point where it intersects $y = 0$.

$x = \frac{-(-1)}{2(2)} = \frac{1}{4}$</td><td>
</td></tr>
<tr><td>Identify the distance from the axis of symmetry to the parabola along $y = 0$.

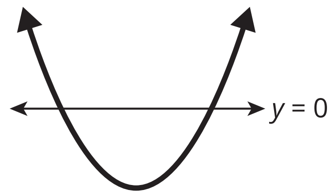
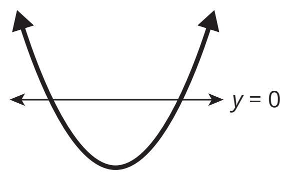
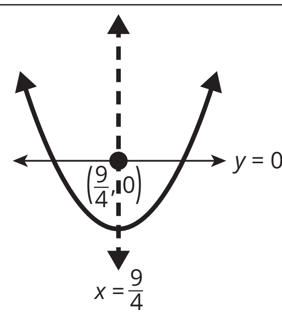
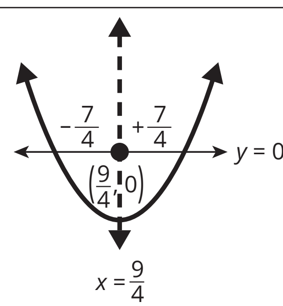
$+\frac{\sqrt{b^2 - 4ac}}{2a} =$

$\frac{\sqrt{(-1)^2 - 4(2)(-15)}}{2(2)} = \frac{\sqrt{121}}{4} = \frac{11}{4}$

$-\frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{11}{4}$</td><td>
</td></tr>
<tr><td>Identify the roots and label them on the graph.

$\frac{1}{4} + \frac{11}{4} = \frac{12}{4} = 3$
$\frac{1}{4} - \frac{11}{4} = -\frac{10}{4} = -\frac{5}{2}$</td><td>
</td></tr>
<tr><td colspan="2">The real roots of $y = 2x^2 - x - 15$ are $x = 3$ and $x = -\frac{5}{2}$.</td></tr>
</table>

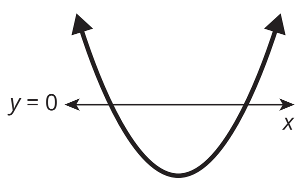
3. Repeat the process to determine the real roots of the equation $y = 2x^2 - 9x + 4$.

Steps	Graph
a. Let $y = 0$ and identify the values of a, b, and c.	
b. Identify the axis of symmetry and label the point where it intersects the x-axis.	
c. Identify the distance from the axis of symmetry to the parabola along $y = 0$ and label the distance on both sides.	
d. Identify the roots and label them on the graph.	
e. Summarize.	

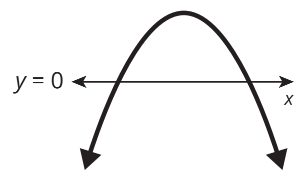
The graphs of quadratic equations are parabolas that have either an absolute maximum or an absolute minimum. A quadratic equation with two real roots crosses the x-axis in two places. A quadratic equation with a double real root, or one unique real root, touches the x-axis but does not cross it.

Quadratic Equations

With Two Real Roots

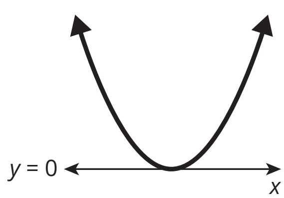


$y = ax^2 + bx + c$
$a > 0$

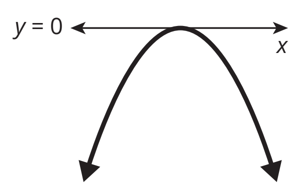


$y = ax^2 + bx + c$
$a < 0$

With Double Real Roots



$y = ax^2 + bx + c$
$a > 0$



$y = ax^2 + bx + c$
$a < 0$

4. Draw and label the following components on each graph in terms of the equation $y = ax^2 + bx + c$.

   a. vertex

   b. axis of symmetry

   c. intersection of the x-axis and line of symmetry

   d. the distance represented by the expression $+\frac{\sqrt{b^2 - 4ac}}{2a}$

   e. the distance represented by the expression $-\frac{\sqrt{b^2 - 4ac}}{2a}$

   f. each root

ACTIVITY 4.3

Using the Quadratic Formula

Let's analyze the structure of the quadratic formula and examine common student mistakes. Consider Michael's work.

1. Michael is determining the exact zeros for $f(x) = x^2 - 14x + 19$. His work is shown.

   a. Identify the error Michael made when determining the zeros.

   b. Determine the correct zeros of the function.

Michael

$$f(x) = x^2 - 14x + 19$$

$$a = 1, b = -14, c = 19$$

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(19)}}{2(1)}$$

$$x = \frac{14 \pm \sqrt{196} - 76}{2}$$

$$x = \frac{14 \pm \sqrt{120}}{2}$$

$$x = \frac{14 \pm \sqrt{30 \cdot 4}}{2}$$

$$x = \frac{14 \pm 2\sqrt{30}}{2}$$

$$x = 7 \pm 2\sqrt{30}$$

"Leave the solutions in exact form" means not to estimate any radical values with rounded decimals.

2. Use the quadratic formula to determine the zeros for each function given. Leave the solutions in exact form and classify them as rational or irrational.

   a. $f(x) = -2x^2 - 3x + 7$

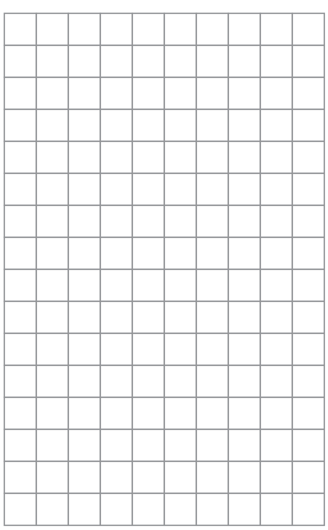
   b. $r(x) = -3x^2 + 19x - 7$

3. Olivia is solving the quadratic equation $x^2 - 7x - 8 = 3$. Her work is shown.

   a. Identify Olivia's error.

   b. Use the quadratic formula correctly to determine the solution to Olivia's quadratic equation. Classify the solutions as rational or irrational.

   c. Use technology to graph each side of the original quadratic equation $x^2 - 7x - 8 = 3$. Sketch your graph. Then, interpret the meaning of the intersection points.



Olivia

$x^2 - 7x - 8 = 3$

$a = 1, b = -7, c = -8$

$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-8)}}{2(1)}$

$x = \frac{7 \pm \sqrt{49 + 32}}{2}$

$x = \frac{7 \pm \sqrt{81}}{2}$

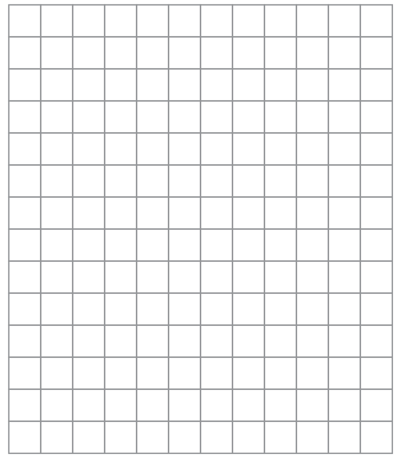
$x = \frac{7 \pm 9}{2}$

$x = \frac{7 + 9}{2}$ or $x = \frac{7 - 9}{2}$

$x = \frac{16}{2} = 8$ or $x = \frac{-2}{2} = -1$

The roots are 8 and -1.

d. Next, rewrite the given quadratic equation so that one side of the equation is equal to zero. Use technology to graph each side of the quadratic equation. Sketch your graph. Interpret the meaning of the intersection points.



Think about . . .
What does it mean to determine the solutions to a quadratic equation? What does it mean to determine the roots of a quadratic equation?

e. Compare the *x*-values of the intersection points from part (c), the *x*-values of the intersection points in part (d), and the solutions using the quadratic formula. What do you notice?

4. Use the quadratic formula to determine the zeros for each function. Round the solutions to the nearest hundredth and classify them as either rational or irrational.

   a. $f(x) = 2x^2 + 10x - 1.02$

   b. $h(x) = 3x^2 - 11x - 2$

Ask Yourself . . .
How can you use graphs to support your reasoning?

5. Reflect on the different quadratic equations you have solved so far in this lesson.

   a. How many real roots does each quadratic equation in this lesson have?

   b. Do all quadratic equations have two real roots? Explain why or why not.

   c. Do you think that a quadratic function could have no real roots? Explain why or why not.

   d. Could a quadratic equation have more than two real roots? Explain why or why not.

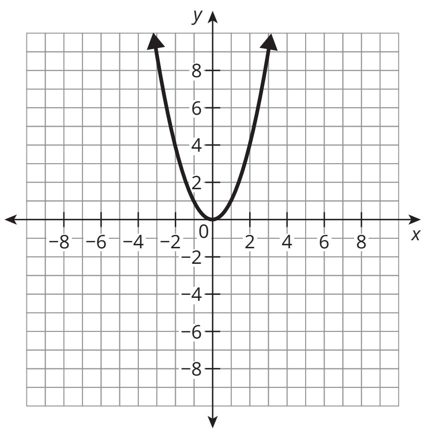
ACTIVITY 4.4

Making Sense of the Discriminant

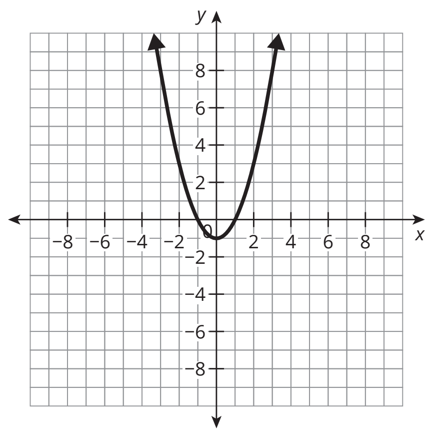
A quadratic function can have one unique real zero, two real zeros, or at times, no real zeros. Let's investigate how the quadratic formula can inform you about different types of zeros.

Consider three quadratic equations and their graphs.

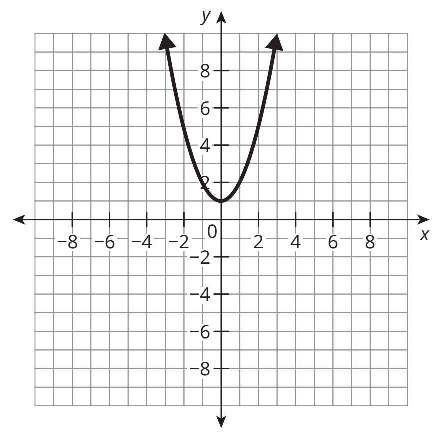
$y = x^2$



$y = x^2 - 1$



$y = x^2 + 1$



Think about . . .
You have analyzed many graphs of quadratic equations. What do you know about the roots of a quadratic equation that touches, but does not intersect, the *x*-axis or intersects the *x*-axis at two points?

1. Use the quadratic formula to solve each quadratic equation. Show your work.

2. What do you notice about the relationship between the number of real roots, the graph, and the results of substituting the values of *a*, *b*, and *c* into the radicand of the quadratic formula?

Because this portion of the formula “discriminates” the number of real zeros, or roots, it is called the **discriminant.**

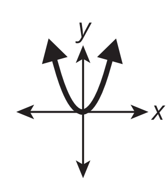
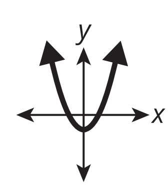
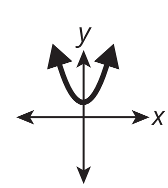
3. Using the discriminant, write an inequality to describe when a quadratic function has each solution.

   a. no real roots/zeros

   b. one unique real root/zero

   c. two unique real roots/zeros

The table shown summarizes the types of solutions for any quadratic equation or function.

		Interpretation of the Solutions		
Equation/ Function	**Solutions**	**Number of Unique Real Zeros**	**Number of *x*-Intercepts**	**Sketch**
$f(x) = x^2$	$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(0)}}{2(1)}$ $= \frac{0 \pm \sqrt{0}}{2}$ $= 0 \pm \sqrt{0}$	1	1	
$g(x) = x^2 - 1$	$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)}$ $= \frac{0 \pm \sqrt{4}}{2}$ $= 0 \pm 1$	2	2	
$h(x) = x^2 + 1$	$x = \frac{-0 \pm \sqrt{0^2 - (4)(1)(1)}}{2(1)}$ $= \frac{0 \pm \sqrt{-4}}{2}$	0	0	

Every quadratic equation with real coefficients has either 2 real roots or 0 real roots. However, when a graph of a quadratic equation has 1 x-intercept, the equation *still* has 2 real roots. In this case, the 2 real roots are considered a double root.

4. Use the discriminant to determine the number of real roots for each equation. Then, solve for the roots/zeros.

   a. $y = 2x^2 + 12x - 2$

   b. $0 = 2x^2 + 12x + 20$

   c. $y = x^2 + 12x + 36$

   d. $y = 3x^2 + 7x - 20$

   e. $y = 4x^2 - 9$

   f. $0 = 9x^2 + 12x + 4$

Talk the Talk

Show Me the Ways

1. Determine the real roots of the quadratic equation $y = 2x^2 + 4x - 6$ using the four methods you learned in this topic.

Factoring	Completing the Square
Using the quadratic formula	Graphing

Ask Yourself . . .
Which tool or strategy is most efficient?

Lesson 4 Assignment

Write

How can you determine the types of solutions when using the quadratic formula?

Remember

The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

On the graph of a quadratic function, $\frac{\pm \sqrt{b^2 - 4ac}}{2a}$ is the distance from $\left(-\frac{b}{2a}, 0\right)$ to each root.

Practice

The formula shown can be used to calculate the distance, s, an object travels in t seconds. In this formula, u represents the initial velocity, and a represents a constant acceleration. Use this formula to answer each question.

$$S = ut + \frac{1}{2}at^2$$

1. Javier is driving 48 miles per hour and is starting to merge onto the highway, so he must increase his speed. He accelerates at a rate of 7 miles per hour for several seconds.

   a. Substitute the initial velocity and constant acceleration into the formula to write an equation to represent the distance Javier travels.

   b. Use the quadratic formula to determine the roots of the equation. What do the roots represent in the context of the problem situation? Explain your reasoning.

Lesson 4 Assignment

2. Daniela is driving her car 32 miles per hour when she passes Koda's house. She then accelerates at a rate of 3 miles per hour for several minutes until she passes the movie theater. Daniela knows that the movie theater is 2.9 miles from Koda's house.

   a. Substitute the initial velocity, constant acceleration, and distance into the formula to write an equation representing the distance Daniela travels.

   b. Use the quadratic formula to determine the roots of the equation you wrote in part (a). What do the roots represent in the context of the problem situation? Explain your reasoning.

3. Use the discriminant to determine the number of real roots for each equation. Then, solve the quadratic equations with real roots.

   a. $4x^2 + 8x - 12 = 0$

   b. $x^2 + 2x - 10 = 0$

Lesson 4 Assignment

c. $9x^2 - 12x + 4 = 0$

d. $3x^2 - 4 = 0$

e. $3x^2 + 2x - 2 = 0$

f. $x^2 - 3x + 5 = 0$

Prepare

1. Determine a linear regression model that best models the data.

x	y
1	32
2	35
3	34
4	35
5	39
6	38
7	40
8	42
9	41

5 Using Quadratic Functions to Model Data

OBJECTIVES

- Use a quadratic function to model data.
- Interpret characteristics of a quadratic function in terms of a problem situation.
- Use graphs of quadratic functions to make estimations and predictions.

You know how to model data with regression.

How can you determine whether a quadratic regression model may best model the data?

Getting Started

That Might Be a Bad Idea. . .

A 12-ounce can of soda was put into a freezer. The table shows the volume of the soda in the can, measured at different temperatures.

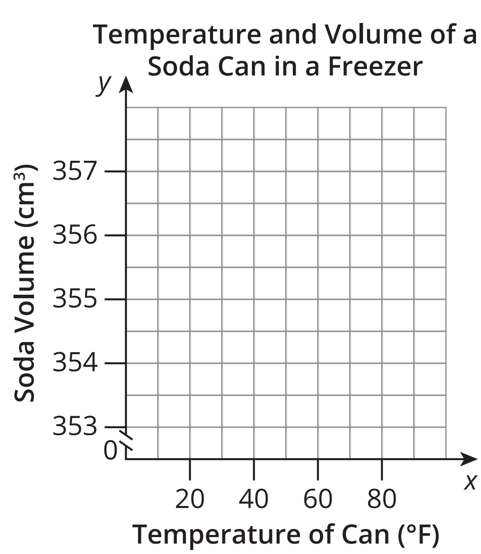
The first step of the modeling process is to notice and wonder. What do you notice about the data? Is there a question it brings to mind that you wonder about?

Temperature of Can (°F)	Soda Volume cm^3
68.0	355.51
50.0	354.98
42.8	354.89
39.2	354.88
35.6	354.89
32.0	354.93
23.0	355.13
14.0	355.54

1. Describe the data distribution.

2. Create a scatterplot of the data. Sketch the plot of points on the coordinate plane shown.

The second step of the modeling process is to organize and mathematize. The scatterplot is a way to organize the data.



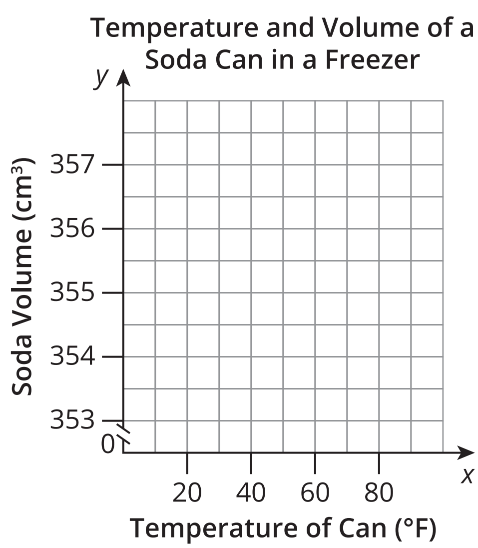
ACTIVITY 5.1

Using Quadratic Functions to Model Data

Let's continue to analyze the data and make some estimations and predictions about the volume of soda at different temperatures.

1. Use technology to calculate the regression model that best models the data in the previous activity. Sketch the graph of the regression model on the coordinate plane on which you created your scatterplot. Explain why the regression model best represents the data.

You can mathematize the data by modeling it with an appropriate regression model.



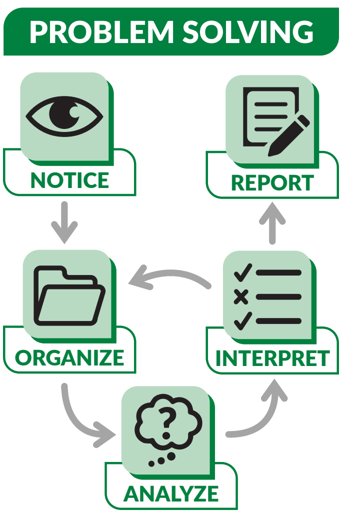
2. State the domain and range of your function. How do they compare to the domain and range of this problem situation?

The third step of the modeling process is to predict and analyze, and the fourth step is to test and interpret. These questions focus on these two steps of the process.

3. Use the regression model to answer each question.

   a. Determine the *y*-intercept and interpret its meaning in terms of this problem situation.

   b. Determine the *x*-intercepts and interpret the meaning of each in terms of this problem situation.



4. Estimate the volume of the soda can when the temperature is:

   a. 20°F.

   b. 60°F.

5. Estimate the temperature when the volume is 356 cm^3.

6. Write a summary of the problem situation, your model of the solution, and any limitations of your model.

ACTIVITY 5.2

Analyzing a Quadratic Model

A local police department is offering special classes for interested high school students this summer. Destiny decides to enroll in an introductory forensic science class. On the first day, Dr. Jackson tells Destiny's class that crime scenes often involve speeding vehicles which leave skid marks on the road as evidence. Taking into account the road surface, weather conditions, the percent grade of the road, and vehicle type, they use the function

$$f(s) = 0.034s^2 + 0.96s - 26.6$$

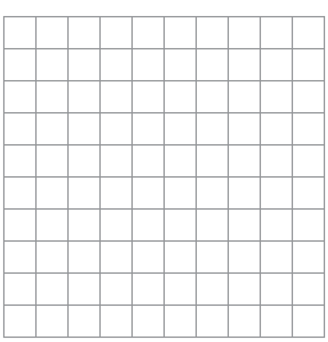
to determine the length in feet of skid marks left by a vehicle based on its speed, s, in miles per hour.

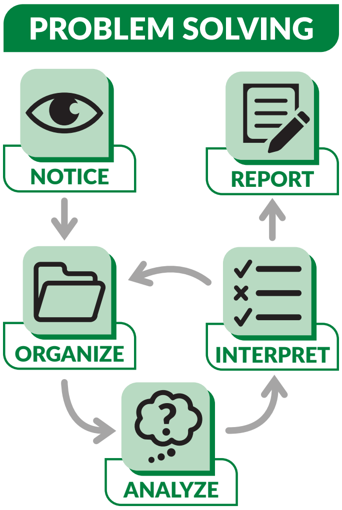
1. Complete the table based on $f(s)$. Label the column titles with the independent and dependent quantities and their units.

25	
30	
45	
55	
60	
75	
90	
100	
110	

2. According to the table, what are the domain and range for the problem situation?

3. Graph the table values and sketch the graph of $f(s)$ on the grid. Label the axes.





Ask Yourself . . .
How do these data differ from the data in the table?

During another class period, Dr. Jackson takes Destiny's class to a mock crime scene to collect evidence.

4. One piece of evidence is a skid mark that is 300 feet long.

   a. Use the graph to estimate the speed of the vehicle that created this skid mark. Explain your process.

   b. Use the equation to estimate the speed of the vehicle that created this skid mark. Show your work.

   c. Does the exact speed seem reasonable for the problem situation?

5. At a second mock crime scene the skid mark is 800 feet long.

   a. Use the equation to predict the speed of the vehicle that created this skid mark. Show your work.

   b. Does the exact speed seem reasonable for the problem situation?

6. Write a report about the length of skid marks left by vehicles and vehicle speeds. Discuss possible factors that would affect the length of the skid marks left by a vehicle and what effect these factors would have on the graph of $f(s)$.

Talk the Talk

Celebrate the Centenarians!

Using data from the US Census Bureau, the table lists the number of Americans, in thousands, who lived to be over 100 years old for the specified years.

Year	Number of Americans (thousands)
1980	32
1990	37
2000	50
2010	53
2020	80

1. Does the data appear to be linear or quadratic?

2. Use graphing technology to determine the regression model of best fit and calculate the correlation coefficient.

3. Estimate how many Americans were over 100 years old in 2005.

4. Predict the number of Americans that will live to be over 100 years old in the year 2030.

5. Predict when there will be 90,000 Americans that will live to be over 100 years old.

Ask Yourself . . .
How can you use quadratic regression models in everyday life?

Lesson 5 Assignment

Write

How would you determine if a quadratic function is the best model for a set of data?

Remember

You can use quadratic regression models to represent real-world situations.

Practice

1. The table shows the percent of public schools with internet access from 1994 to 2005. (The largest growth years are shown in the table.)

Year	Percent of Public Schools with Internet Access
1994	35
1995	50
1996	65
1997	78
1998	89
1999	95
2000	98
2001	99
2002	99
2003	100
2005	100

   a. Predict whether a linear or quadratic regression model will best fit the data. Explain your reasoning.

   b. Create a scatterplot of the data.

   c. Does your scatterplot change or support your answer to part (a)? Explain your reasoning.

Lesson 5 Assignment

2. Fernando thinks a quadratic regression model is the best fit the data.

   a. Calculate the quadratic regression model of the data. Do you agree with Fernando? Explain your reasoning.

   b. Year 2004 is missing from the data. Calculate the percent of public schools with internet access in 2004. Does your answer make sense in terms of the problem situation? Explain your reasoning.

   c. Calculate the percent of public schools with internet access in 2020. Does your answer make sense in terms of the problem situation? Explain your reasoning.

   d. What are the *x*-intercepts and what do they mean in terms of the problem situation?

Lesson 5 Assignment

e. In what year does the percent of public schools with internet access begin to decline? Explain how you determined your answer.

f. Do you think it is likely that the percent of public schools with internet access will decline? Explain your reasoning.

TOPIC 3 SELF-REFLECTION

Name: ______________________________

Solving Quadratic Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents I am **building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill.**

I can demonstrate an understanding of the standards in the *Solving Quadratic Equations* topic by:

TOPIC 3: *Solving Quadratic Equations*	Beginning of Topic	Middle of Topic	End of Topic
solving quadratic equations by the most efficient method: inspection, properties of equality and square roots, graphically, factoring, completing the square, or using the quadratic formula.	◯	◯	◯
recognizing special products, including the difference of two squares, and using them to factor polynomials.	◯	◯	◯
factoring trinomials, when possible, including perfect square trinomials.	◯	◯	◯
completing the square to rewrite a quadratic expression in vertex form.	◯	◯	◯
explaining that the quadratic formula is derived by completing the square of the standard form of a quadratic equation $y = ax^2 + bx + c$.	◯	◯	◯
understanding that the quadratic formula, $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, represents axis of symmetry plus or minus the distance to the parabola.	◯	◯	◯
using technology, determining the quadratic regression model that provides a reasonable fit to data to estimate solutions and making predictions for real-world problems.	◯	◯	◯

continued on the next page

TOPIC 3 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Solving Quadratic Equations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 3 SUMMARY

Solving Quadratic Equations Summary

NEW KEY TERMS

- principal square root
- roots [raíces]
- double root [raíz doble]
- zero product property [propiedad del producto cero]
- completing the square
- quadratic formula [fórmula cuadrática]
- discriminant [discriminante]

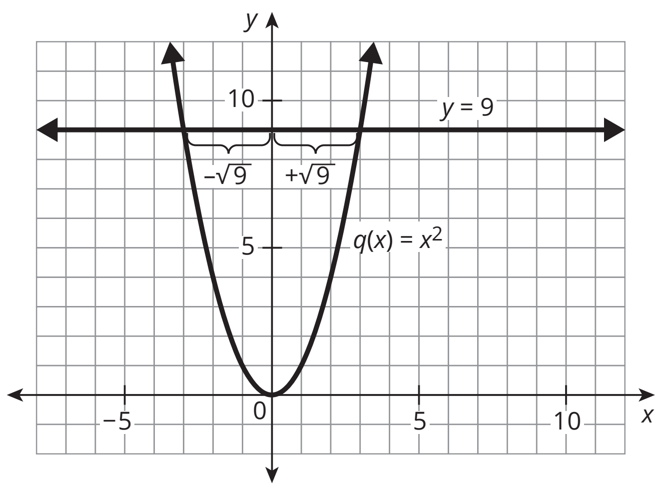
LESSON 1 Representing Solutions to Quadratic Equations

A quadratic function is a function of degree 2 because the greatest power for any of its terms is 2. This means that it has at most 2 zeros, or at most 2 solutions, at $y = 0$.

The two solutions of a quadratic function can be represented as square roots of numbers. Every positive number has two square roots, a positive square root, which is also called the **principal square root**, and a negative square root. To solve the equation $x^2 = 9$, take the square root of both sides of the equation.

$$\sqrt{x^2} = \sqrt{9}$$
$$x = \pm 3$$

You can solve $x^2 = 9$ on a graph by finding the points of intersection between $y = x^2$ and $y = 9$. The solutions are both 3 units from the axis of symmetry, $x = 0$.



The x-intercepts of a graph of a quadratic function are called the *zeros* of the quadratic function. The zeros are called the **roots** of the quadratic equation.

The quadratic function $q(x) = x^2$ has two solutions at $y = 0$; therefore, it has 2 zeros: $x = +\sqrt{0}$ and $x = -\sqrt{0}$. These two zeros of the function, or roots of the equation, are the same number, 0, so the function $q(x) = x^2$ is said to have a **double root**.

When you encounter solutions that are not perfect squares, you can either determine the approximate value of the radical or rewrite it in an equivalent radical form.

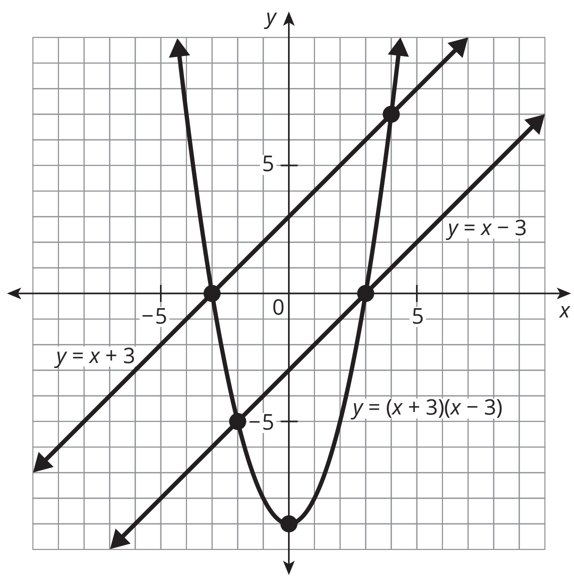
To approximate a square root, determine the perfect square that is closest to, but less than, the given value and the perfect square that is closest to, but greater than, the given value. You can use these square roots to approximate the square root of the given number.

For example, the approximate value of $\sqrt{40}$ falls between $\sqrt{36}$, or 6, and $\sqrt{49}$, or 7. Since $6.3^2 = 39.69$ and $6.4^2 = 40.96$, the approximate value of $\sqrt{40}$ is 6.3.

To rewrite a square root in equivalent radical form, first rewrite the product of the radicand to include any perfect square factors. Then, extract the square roots of those perfect squares.

$$\begin{aligned}\sqrt{27} &= \sqrt{9 \cdot 3}\\ &= \sqrt{9} \cdot \sqrt{3}\\ &= 3\sqrt{3}\end{aligned}$$

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$. In factored form, r_1 and r_2 represent the x-intercepts of the graph of the function. The x-intercepts of the graph of the quadratic function $f(x) = ax^2 + bx + c$ and the zeros of the function are the same as the roots of the equation $ax^2 + bx + c = 0$.



To write a quadratic function in factored form, first determine the zeros of the function $f(x) = x^2 - 9$, set the trinomial expression equal to 0, and solve for x.

$$\begin{aligned}0 &= x^2 - 9\\ 9 &= x^2\\ \sqrt{9} &= \sqrt{x^2}\\ \pm 3 &= x\end{aligned}$$

You can then use the zeros to write the function in factored form, $f(x) = (x + 3)(x - 3)$.

The **zero product property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero. You can see from the graph that the zeros of the function $f(x) = x^2 - 9$ occur where either $y = x + 3$ or $y = x - 3$ are zero.

LESSON 2 Solutions to Quadratic Equations in Vertex Form

The solutions to any quadratic equation are located on the parabola, equidistant from the axis of symmetry.

A quadratic function in vertex form $f(x) = a(x - h)^2 + k$ is translated horizontally h units, dilated vertically by the factor a, and translated vertically k units.

For the equation $y = (x - c)^2$, the solutions can be represented by $c \pm \sqrt{y}$.
For the equation $y = a(x - c)^2$, the solutions can be represented by $c \pm \sqrt{\frac{y}{a}}$.
For the equation $y = a(x - c)^2 + d$, the solutions can be represented by $c \pm \sqrt{\frac{y - d}{a}}$.

LESSON 3 Factoring and Completing the Square

You can factor trinomials by rewriting them as the product of two linear expressions.

For example, to factor the trinomial $x^2 + 10x + 16$, determine the factor pairs of the constant term. The factors of 16 are (1)(16), (2)(8), and (4)(4). Then, determine the pair whose sum is the coefficient of the middle term, 10.

	x	8
x	x^2	$8x$
2	$2x$	16

The sum of $2x$ and $8x$ is $10x$. So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

You can use factoring and the zero product property to solve quadratics in the form $y = ax^2 + bx + c$.

For example, you can solve the quadratic equation $x^2 - 4x = -3$.

$$x^2 - 4x = -3$$
$$x^2 - 4x + 3 = -3 + 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

$$(x - 3) = 0 \quad \text{or} \quad (x - 1) = 0$$
$$x - 3 + 3 = 0 + 3 \quad \text{or} \quad x - 1 + 1 = 0 + 1$$
$$x = 3 \quad \text{or} \quad x = 1$$

For a quadratic function that has zeros but cannot be factored, there is another method for solving the quadratic equation. **Completing the square** is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.

For example, you can calculate the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$.	$x^2 - 4x + 2 - 2 = 0 - 2$ $x^2 - 4x = -2$
Complete the square and rewrite this as a perfect square trinomial.	
Determine the constant term that would complete the square.	$x^2 - 4x + ? = -2 + ?$
Add this term to both sides of the equation.	$x^2 - 4x + 4 = -2 + 4$ $x^2 - 4x + 4 = 2$
Factor the left side of the equation.	$(x - 2)^2 = 2$
Determine the square root of each side of the equation.	$\sqrt{(x - 2)^2} = \sqrt{2}$ $(x - 2) = \pm\sqrt{2}$
Set the factor of the perfect square trinomial equal to each square root of the constant and solve for x.	$x - 2 = \sqrt{2}$ or $x - 2 = -\sqrt{2}$ $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$ $x \approx 3.41$ or $x \approx 0.59$

The roots are approximately 3.41 and 0.59.

Completing the square can also be used to identify the axis of symmetry and the vertex of any quadratic function written in standard form.

When a function is written in standard form, $ax^2 + bx + c$, the axis of symmetry is $x = -\frac{b}{2a}$.

Given a quadratic equation in the form $y = ax^2 + bx + c$, the vertex of the function is located at $x = -\frac{b}{2a}$ and $y = c - \frac{b^2}{4a}$.

LESSON 4

The Quadratic Formula

You can use the **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

For example, given the function $f(x) = 2x^2 - 4x - 3$ we can identify the values of a, b, and c.

$$a = 2; b = -4; c = -3$$

Then, we use the quadratic formula to solve.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x \approx \frac{4 + 6.325}{4} \approx 2.581 \quad \text{or}$$

$$x \approx \frac{4 - 6.325}{4} \approx -0.581$$

The roots are approximately 2.581 and (−0.581).

A quadratic function can have one real zero, two real zeros, or at times, no real zeros.

You can use the part of the quadratic formula underneath the square root symbol to identify the number of real zeros or roots. Because this portion of the formula "discriminates" the number of real zeros or roots, it is called the **discriminant**.

When the discriminant is positive, the quadratic has two real roots. If the discriminant is negative, the quadratic has no real roots. When the discriminant is 0, the quadratic has a double real root.

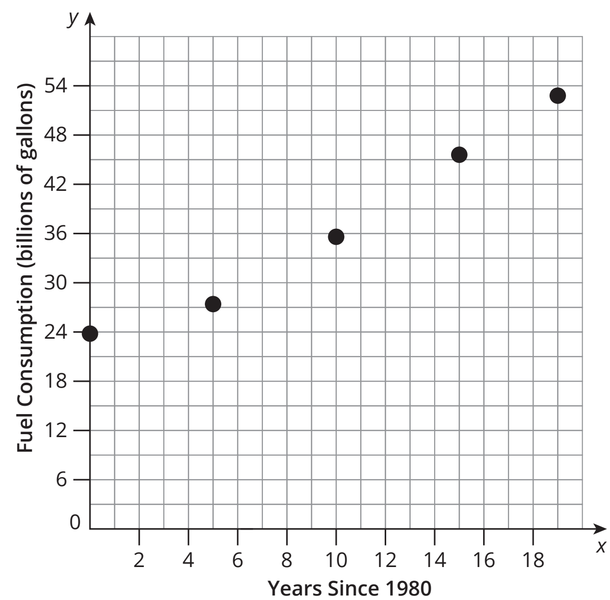
You can also use the discriminant to describe the nature of the roots. If the discriminant is a perfect square, then the roots are rational. If the discriminant is not a perfect square, then the roots are irrational.

LESSON 5 Using Quadratic Functions to Model Data

Quadratic regression models can be used to represent real-world situations and make estimations and predictions.

For example, as vans, trucks, and SUVs have increased in popularity, the fuel consumption of these types of vehicles has also increased.

Years Since 1980	Fuel Consumption (billions of gallons)
0	23.8
5	27.4
10	35.6
15	45.6
19	52.8



The quadratic regression model that best fits the data is $y = 0.0407x^2 + 0.809x + 23.3$. The r^2-value for the quadratic regression fit is 0.996. Just as with linear and exponential regressions, the model can be used to make predictions for the data.

For example, you can predict the fuel consumption in the year 2020 by substituting $x = 40$ into the regression model.

$$y = 0.0407(40)^2 + 0.809(40) + 23.3$$
$$y \approx 121$$

In 2020, fuel consumption was about 121 billion gallons.

Math Glossary

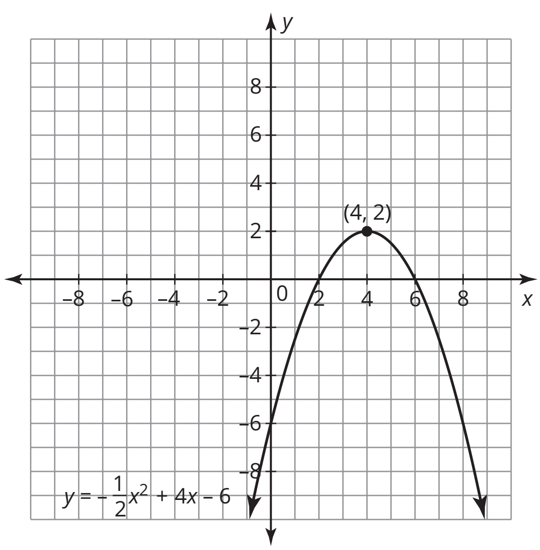
A

absolute maximum

A function has an absolute maximum if there is a point that has a y-coordinate that is greater than the y-coordinates of every other point on the graph.

Example

The ordered pair (4, 2) is the absolute maximum of the graph of the function $f(x) = -\frac{1}{2}x^2 + 4x - 6$.

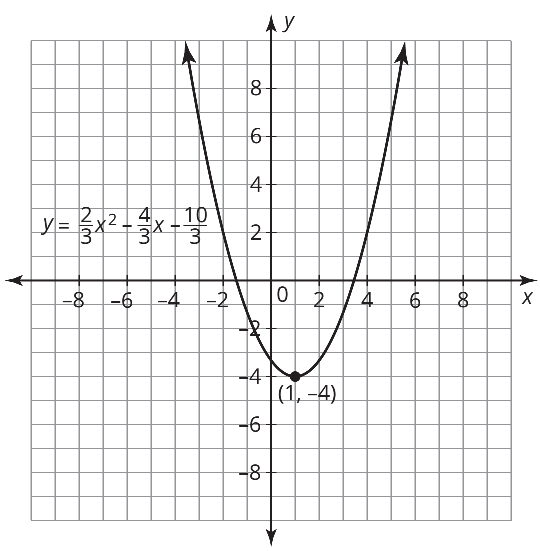


absolute minimum

A function has an absolute minimum if there is a point that has a y-coordinate that is less than the y-coordinates of every other point on the graph.

Example

The ordered pair (1, −4) is the absolute minimum of the graph of the function $y = \frac{2}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$.



argument of a function

The argument of a function is the variable on which the function operates.

Example

In the function $f(x + 5) = 32$, the argument is $x + 5$.

arithmetic sequence

An arithmetic sequence is a sequence of numbers in which the difference between any two consecutive terms is a constant.

Example

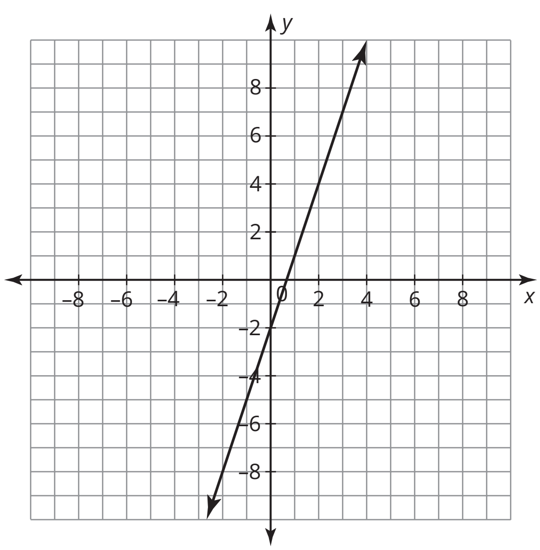
The sequence 1, 3, 5, 7 is an arithmetic sequence with a common difference of 2.

average rate of change

Another name for the slope of a linear function is average rate of change. The formula for the average rate of change is $\frac{f(t) - f(s)}{t - s}$.

Example

The average rate of change of the function shown is 3.

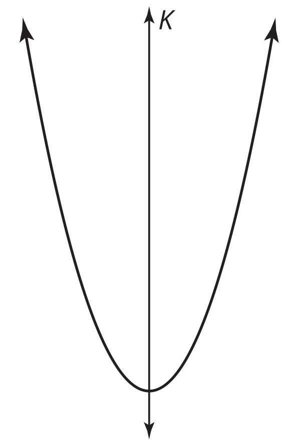


axis of symmetry

The axis of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images.

Example

Line *K* is the axis of symmetry of this parabola.



B

base

The base of a power is the expression that is used as a factor in the repeated multiplication.

binomial

Polynomials with exactly two terms are binomials.

Example

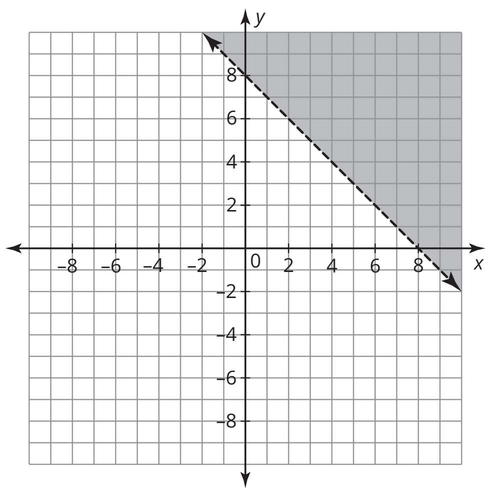
The polynomial $3x + 5$ is a binomial.

boundary line

A boundary line, determined by the inequality in a linear inequality, divides the plane into two half-planes and the inequality symbol indicates which half-plane contains all the solutions.

Example

For the linear inequality $y > -x + 8$, the boundary line is a dashed line because no point on that line is a solution.



C

causation

Causation is when one event affects the outcome of a second event.

centroid

The centroid is a point in which *x*-value is the mean of all the *x*-values of the points on the scatterplot and its *y*-value is the mean of all the *y*-values of the points on the scatterplot.

Example

For the data points (1, 3), (1, 7), (2, 6), (3, 5), and (3, 4), the centroid is (2, 5).

closed (closure)

When an operation is performed on any of the numbers in a set and the result is a number that is also in the same set, the set is said to be closed (or to have closure) under that operation.

Example

The set of whole numbers is closed under addition. The sum of any two whole numbers is always another whole number.

coefficient of determination

The coefficient of determination measures how well the graph of a regression fits the data. It is calculated by squaring the correlation coefficient and represents the percentage of variation of the observed values of the data points from their predicted values.

Example

The correlation coefficient for a data set is -0.9935. The coefficient of determination for the same data set is approximately 0.987, which means 98.7% of the data values should fall on the graph.

common difference

The difference between any two consecutive terms in an arithmetic sequence is called the common difference. It is typically represented by the variable *d*.

Example

The sequence 1, 3, 5, 7 is an arithmetic sequence with a common difference of 2.

common ratio

The ratio between any two consecutive terms in a geometric sequence is called the common ratio. It is typically represented by the variable *r*.

Example

The sequence 2, 4, 8, 16 is a geometric sequence with a common ratio of 2.

common response

A common response is when a variable other than the ones measured cause the same result as the one observed in the experiment.

completing the square

Completing the square is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

compound interest

In a compound interest account, the balance is multiplied by the same amount at each interval.

Example

Sonya opens a savings account with $100. She earns $4 in compound interest the first year. The compound interest y is found by using the equation $y = 100(1 + 0.04)^t$, where t is the time in years.

concave down

A graph that opens downward is identified as being concave down.

concave up

A graph that opens upward is identified as being concave up.

confounding variable

A confounding variable is when there are other variables in an experiment that are unknown or unobserved.

conjecture

A conjecture is a mathematical statement that appears to be true but has not been formally proven.

consistent systems

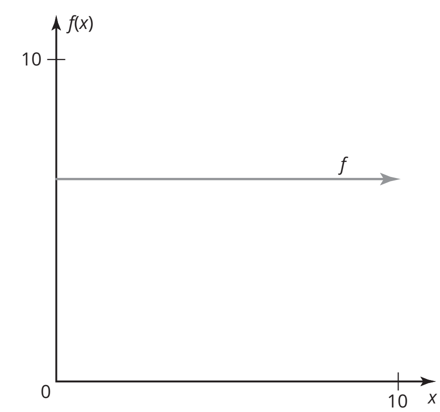
Systems that have one or many solutions are called consistent systems.

constant function

If the dependent variable of a function does not change or remains constant over the entire domain, then the function is called a constant function.

Example

The function shown is a constant function.



constraints

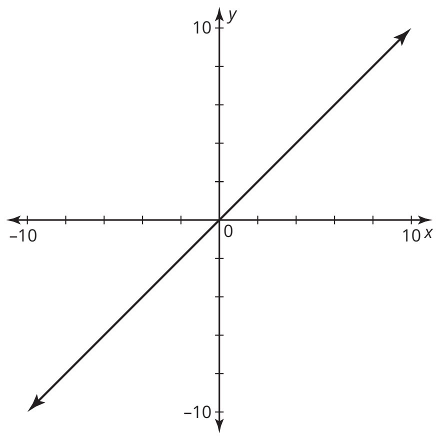
In a system of linear inequalities, the inequalities are known as constraints because the values of the expressions are "constrained" to lie within a certain region on the graph.

continuous graph

A continuous graph is a graph of points that are connected by a line or smooth curve on the graph. Continuous graphs have no breaks.

Example

The graph shown is a continuous graph.



correlation

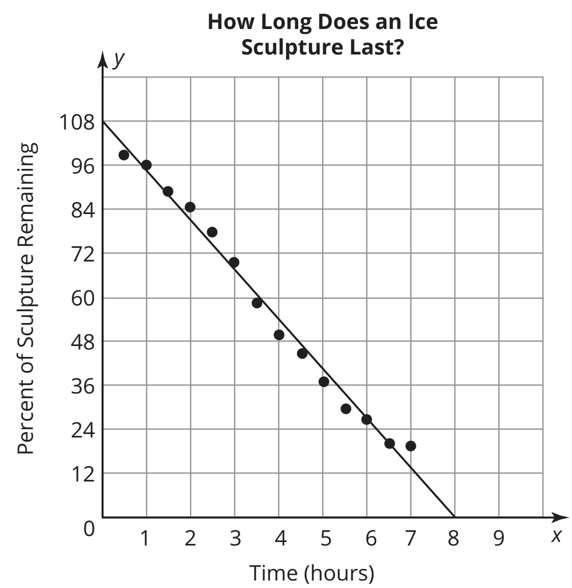
A measure of how well a regression fits a set of data is called a correlation.

correlation coefficient

The correlation coefficient is a value between −1 and 1, which indicates how close the data are to the graph of the regression function. The closer the correlation coefficient is to −1 or 1, the stronger the relationship is between the two variables. The variable *r* is used to represent the correlation coefficient.

Example

The correlation coefficient for these data is −0.9935. The value is negative because the equation has a negative slope. The value is close to −1 because the data are very close to the graph of the equation of the line.



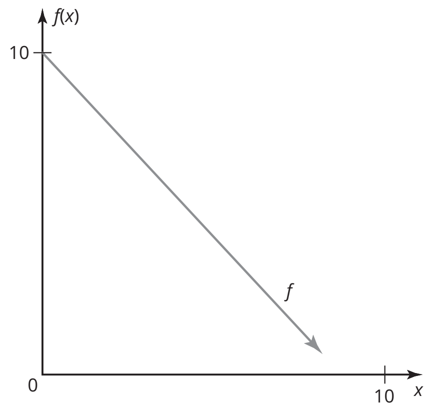
D

decreasing function

If a function decreases across the entire domain, then the function is called a decreasing function.

Example

The function shown is a decreasing function.



degree

The degree of a polynomial is the greatest variable exponent in the expression.

degree of a polynomial

The greatest exponent for any variable term in a polynomial determines the degree of the polynomial.

Example

The polynomial $2x^3 + 5x^2 - 6x + 1$ has a degree of 3.

dependent quantity

When one quantity is determined by another in a problem situation, it is said to be the dependent quantity.

Example

In the relationship between driving time and distance traveled, distance is the dependent quantity, because distance depends on the driving time.

difference of two squares

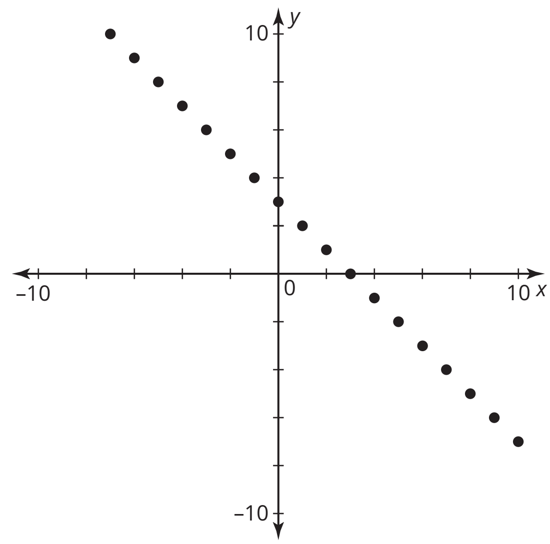
The difference of two squares is an expression in the form $a^2 - b^2$ that can be factored as $(a + b)(a - b)$.

discrete graph

A discrete graph is a graph of isolated points.

Example

The graph shown is a discrete graph.



discriminant

The discriminant is the radicand expression in the quadratic formula which "discriminates" the number of real roots of a quadratic equation.

Example

The discriminant in the quadratic formula is the expression $b^2 - 4ac$.

domain

The domain is the set of input values in a relation.

Example

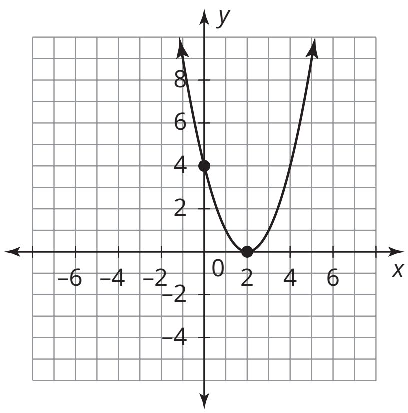
The domain of the function $y = 2x$ is the set of all real numbers.

double root

The root of an equation indicates where the graph of the equation crosses the x-axis.
A double root occurs when the graph just touches the x-axis but does not cross it.

Example

The quadratic equation $y = (x - 2)^2$ has a double root at $x = 2$.



E

explicit formula

An explicit formula of a sequence is a formula for calculating the value of each term of a sequence using the term's position in the sequence. The explicit formula for an arithmetic sequence is $a_n = a_1 + d(n - 1)$. The explicit formula for a geometric sequence is
$g_n = g_1 \cdot r^{n-1}$.

Example

The sequence 1, 3, 5, 7, 9, . . . can be described by the rule $a_n = 2n - 1$ where n is the position of the term. The fourth term of the sequence
a_4 is 2(4) − 1, or 7.

exponent

The exponent of a power is the number of times that the base is used as a factor in the repeated multiplication.

exponential decay function

An exponential decay function is an exponential function with a b-value greater than 0 and less than 1 and is of the form $y = a(1 - r)^x$, where r is the rate of decay.

Example

Greenville has a population of 7000. Its population is decreasing at a rate of 1.75%. The exponential decay function that models this situation is $f(x) = 7000 \cdot 0.9825^x$.

exponential functions

The family of exponential functions includes functions of the form $f(x) = ab^x$, where a and b are real numbers, and b is greater than 0 but is not equal to 1.

Example

The function $f(x) = 2^x$ is an exponential function.

exponential growth function

An exponential growth function is an exponential function with a b-value greater than 1 and is of the form $y = a(1 + r)^x$, where r is the rate of growth.

Example

Blueville has a population of 7000. Its population is increasing at a rate of 1.4%. The exponential growth function that models this situation is
$f(x) = 7000 \cdot 1.014^x$.

extract the square root

To extract a square root, solve an equation of the form $a^2 = b$ for a.

extrapolation

To make predictions for values of x that are outside of the data set is called extrapolation.

F

Factor theorem

The Factor theorem states that a polynomial function $p(x)$ has $x - r$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$.

factored form

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

Example

The function $h(x) = x^2 - 8x + 12$ written in factored form is $(x - 6)(x - 2)$.

finite sequence

If a sequence terminates, it is called a finite sequence.

Example

The sequence 22, 26, 30 is a finite sequence.

first differences

First differences are the values determined by subtracting consecutive output values in a table when the input values have an interval of 1.

Example

	Time (minutes)	Height (feet)	First Differences
	0	0	
$1 - 0 = 1$	1	1800	$1800 - 0 = 1800$
$2 - 1 = 1$	2	3600	$3600 - 1800 = 1800$
$3 - 2 = 1$	3	5400	$5400 - 3600 = 1800$

function

A function is a relation that assigns to each element of the domain exactly one element of the range.

Example

The equation $y = 2x$ is a function. Every value of x has exactly one corresponding y-value.

function family

A function family is a group of functions that share certain characteristics.

Example

Linear functions and exponential functions are examples of function families.

function notation

Function notation is a way of representing functions algebraically.

Example

In the function $f(x) = 0.75x$, f is the name of the function, x represents the domain, and $f(x)$ represents the range.

G

geometric sequence

A geometric sequence is a sequence of numbers in which the ratio between any two consecutive terms is a constant.

Example

The sequence 2, 4, 8, 16 is a geometric sequence with a common ratio of 2.

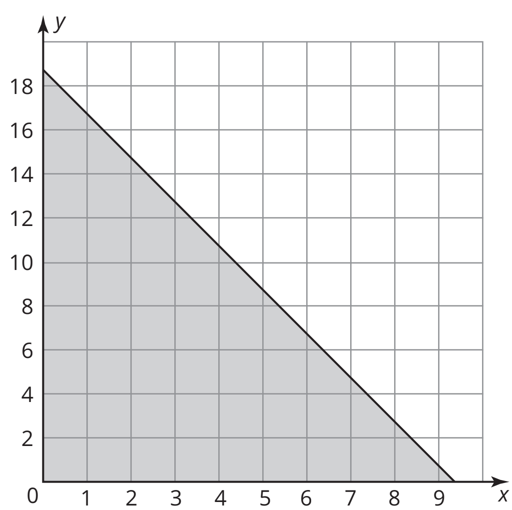
H

half-plane

The graph of a linear inequality is a half-plane, or half of a coordinate plane.

Example

The shaded portion of the graph is a half-plane.

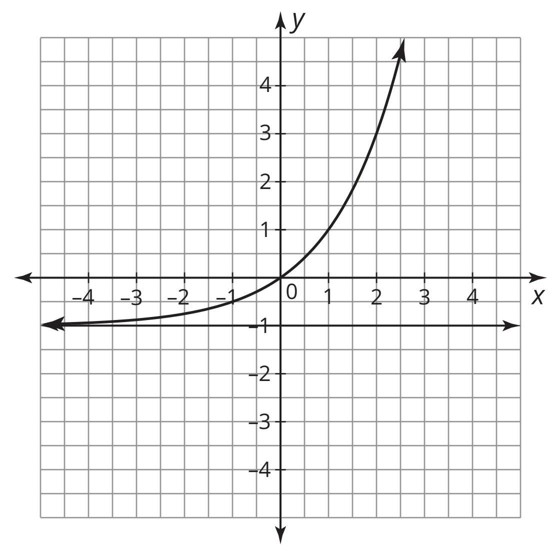


horizontal asymptote

A horizontal asymptote is a horizontal line that a function gets closer and closer to but never intersects.

Example

The graph shows a horizontal asymptote at $y = -1$.



I

inconsistent systems

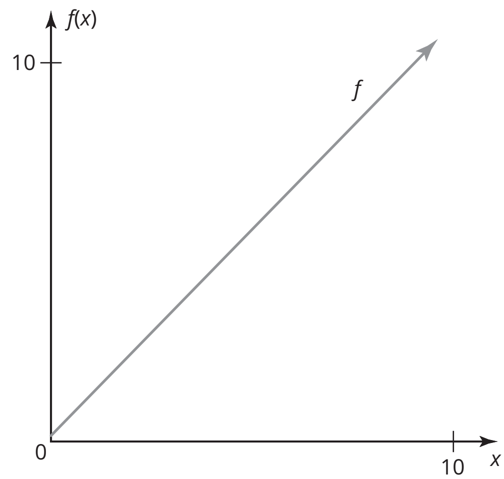
Systems with no solution are called inconsistent systems.

increasing function

If a function increases across the entire domain, then the function is called an increasing function.

Example

The function shown is an increasing function.



independent quantity

The quantity that the dependent quantity depends upon is called the independent quantity.

Example

In the relationship between driving time and distance traveled, driving time is the independent quantity, because it does not depend on any other quantity.

infinite sequence

If a sequence continues on forever, it is called an infinite sequence.

Example

The sequence 22, 26, 30, 34 . . . is an infinite sequence.

infinite solutions

An equation with infinite solutions means that any value for the variable makes the equation true.

Example

The equation $2x + 1 = 2x + 1$ has infinite solutions.

interpolation

Using a linear regression to make predictions within the data set is called interpolation.

L

leading coefficient

The leading coefficient of a polynomial is the numeric coefficient of the term with the greatest power.

Example

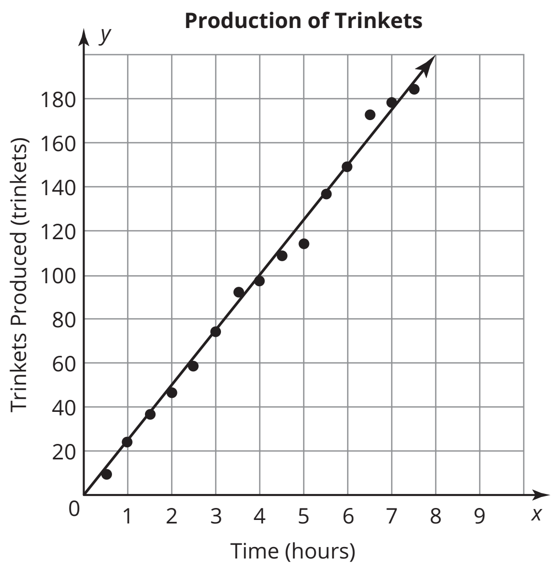
In the polynomial $-7x^2 + x + 25$, the value -7 is the leading coefficient.

Least Squares Method

The Least Squares Method is a method that creates a regression line for a scatterplot that has two basic requirements: 1) the line must contain the centroid of the data set, and 2) the sum of the squares of the vertical distances from each given data point is at a minimum with the line.

Example

The regression line shown was created using the Least Squares Method.



linear combinations method

The linear combinations method is a process used to solve a system of equations by adding two equations together, resulting in an equation with one variable.

Example

Solve the following system of equations by using the linear combinations method:

$$\begin{cases} 6x - 5y = 3 \\ 2x + 2y = 12 \end{cases}$$

First, multiply the second equation by -3. Then, add the equations and solve for the remaining variable. Finally, substitute $y = 3$ into the first equation and solve for x. The solution of the system is (3, 3).

linear functions

The family of linear functions includes functions of the form $f(x) = ax + b$, where a and b are real numbers.

Example

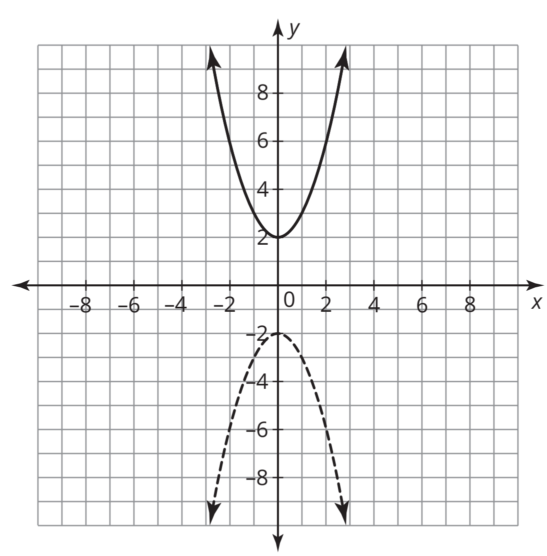
The function $f(x) = 3x + 2$ is a linear function.

line of reflection

A line of reflection is the line that the graph is reflected across.

Example

The graph of $y = x^2 + 2$ was reflected across the line of reflection, $y = 0$.



literal equation

Literal equations are equations in which the variables represent specific measures.

Example

The equations $I = Prt$ and $A = lw$ are literal equations.

M

mathematical modeling

Mathematical modeling is explaining patterns in the real world based on mathematical ideas.

monomial

Polynomials with only one term are monomials.

Example

The expressions $5x$, 7, $-2xy$, and $13x^3$ are monomials.

N

necessary condition

A correlation is a necessary condition for causation, meaning that for one variable to cause another, they must be correlated.

no solution

An equation with no solution means that there is no value for the variable that makes the equation true.

Example

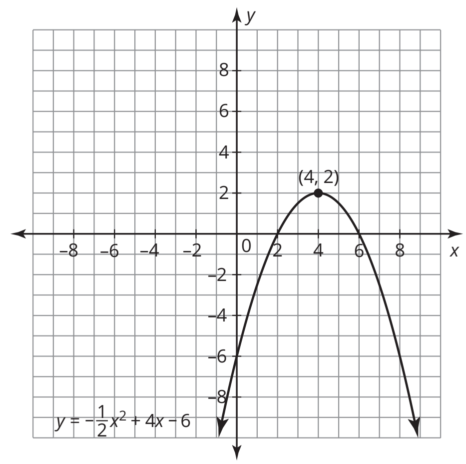
The equation $2x + 1 = 2x + 3$ has no solution.

P

parabola

The shape that a quadratic function forms when graphed is called a parabola. A parabola is a smooth curve with reflectional symmetry.

Example



parent function

A parent function is the simplest function of its type.

Examples

The parent linear function is $f(x) = x$.
The parent exponential function is $g(x) = 2^x$.
The parent quadratic function is $h(x) = x^2$.

perfect square trinomial

A perfect square trinomial is an expression in the form $a^2 + 2ab + b^2$ or in the form $a^2 - 2ab + b^2$.

point-slope form

The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line.

polynomial

A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients.

Example

The expression $3x^3 + 5x - 6x + 1$ is a polynomial.

polynomial long division

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.

Example

Polynomial Long Division

$(2x^2 + 5x - 12) \div (x + 4)$

or

$\frac{2x^2 + 5x - 12}{x + 4}$

$$\begin{array}{r} \text{Ⓐ}\,2x \;\text{Ⓓ}\!-3 \\ x+4\,\overline{)\,2x^2 + 5x - 12} \\ \text{Ⓑ}\,\underline{-(2x^2 + 8x)}\downarrow \;\text{Ⓒ} \\ -3x - 12 \\ \underline{-(-3x - 12)} \\ \text{Remainder } 0 \end{array}$$

A. Divide $\frac{2x^2}{x} = 2x$.

B. Multiply $2x(x + 4)$, and then subtract.

C. Bring down -12.

D. Divide $\frac{-3x}{x} = -3$.

E. Multiply $-3(x + 4)$, and then subtract.

power

A power has a *base* and an *exponent*.

principal square root

The principal square root is a positive square root of a number.

Q

quadratic formula

The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c$, where a, b, and c represent real numbers and $a \neq 0$.

quadratic functions

The family of quadratic functions includes functions of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, and a is not equal to 0.

Examples

The equations $y = x^2 + 2x + 5$ and $y = -4x^2 - 7x + 1$ are quadratic functions.

R

range

The range is the set of output values in a relation.

Example

The range of the function $y = x^2$ is the set of all numbers greater than or equal to zero.

recursive formula

A recursive formula expresses each new term of a sequence based on the preceding term in the sequence. The recursive formula for an arithmetic sequence is $a_n = a_{n-1} + d$. The recursive formula for a geometric sequence is $g_n = g_{n-1} \cdot r$.

Example

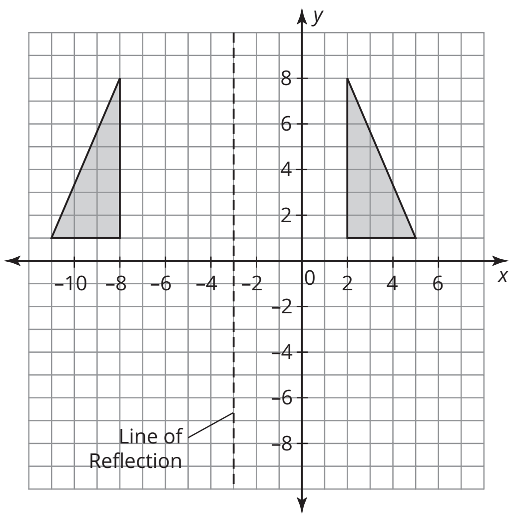
The formula $a_n = a_{n-1} + 2$ is a recursive formula. Each successive term is calculated by adding 2 to the previous term. If $a_1 = 1$, then $a_2 = 1 + 2 = 3$.

reflection

A reflection of a graph is a mirror image of the graph about a line of reflection.

Example

The triangle on the right is a reflection of the triangle on the left.



regression function

On a scatterplot, a regression function is a mathematical model that can be used to predict the values of a dependent variable based upon the values of an independent variable.

relation

A relation is the mapping between a set of input values called the domain and a set of output values called the range.

Example

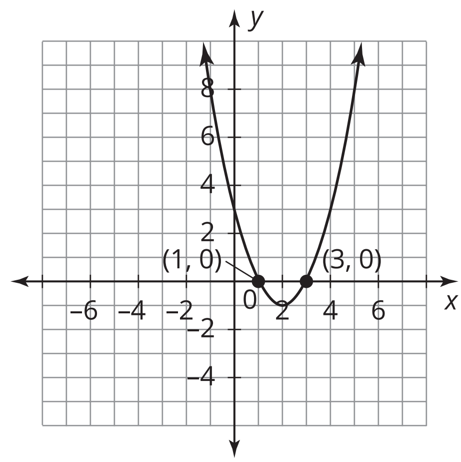
The set of points {(0, 1), (1, 8), (2, 5), (3, 7)} is a relation.

root (roots)

The root or roots of an equation indicate where the graph of the equation crosses the *x*-axis.

Example

The roots of the quadratic equation $x^2 - 4x + 3 = 0$ are $x = 3$ and $x = 1$.



S

second differences

Second differences are the differences between consecutive values of the first differences.

Example

x	*y*	First Differences	Second Differences
–3	–5	5	–2
–2	0	3	–2
–1	3	1	–2
0	4	–1	–2
1	3	–3	–2
2	0	–5	
3	–5		

sequence

A sequence is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects.

Example

The numbers 1, 1, 2, 3, 5, 8, 13 form a sequence.

simple interest

In a simple interest account, the interest earned at the end of each interval is a percent of the starting balance (also known as the principal).

Example

Tonya deposits $200 in a 3-year certificate of deposit that earns 4% simple interest. The amount of interest that Tonya earns can be found using the simple interest formula.

$$I = (200)(0.04)(3)$$

$$I = 24$$

Tonya earns $24 in interest.

solution

The solution to an equation is any value for the variable that makes the equation a true statement.

Example

The solution of the equation $3x + 4 = 25$ is 7 because 7 makes the equation true: $3(7) + 4 = 25$, or $25 = 25$.

solution set of a system of linear inequalities

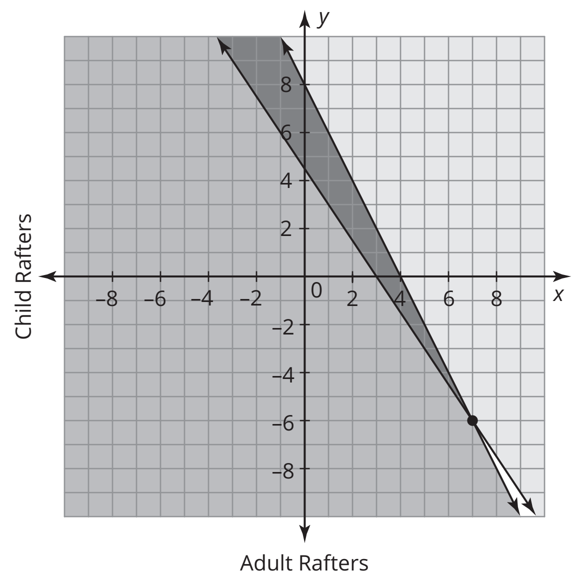
The solution set of a system of linear inequalities is the intersection of the solutions to each inequality. Every point in the intersection region satisfies all inequalities in the system.

Example

The solution set of this system of linear inequalities ...

$$\begin{cases} 200a + 100c \leq 800 \\ 75(a - 1) + 50c \geq 150 \end{cases}$$

... is shown by the shaded region, which represents the intersection of the solutions to each inequality.



solve an inequality

To solve an inequality means to determine the values of the variable that make the inequality true.

Example

The inequality $x + 5 > 6$ can be solved by subtracting 5 from each side of the inequality. The solution is $x > 1$. Any number greater than 1 will make the inequality $x + 5 > 6$ true.

standard form of a linear equation

The standard form of a linear equation is $Ax + By = C$, where A, B, and C are integers and A and B are not both zero.

standard form of a quadratic function

A quadratic function written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is in standard form.

Example

The function $f(x) = -5x^2 - 10x + 1$ is written in standard form.

sufficient condition

A correlation is not a sufficient condition for causation, meaning that a correlation between two variables is not enough to establish that one variables causes another.

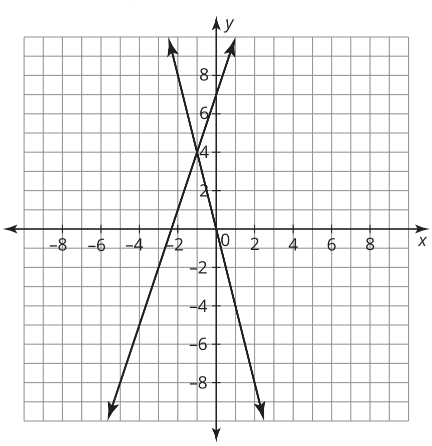
system of linear equations

When two or more linear equations define a relationship between quantities, they form a system of linear equations.

Example

The equations $y = 3x + 7$ and $y = -4x$ are a system of linear equations.

$$\begin{cases} y = 3x + 7 \\ y = -4x \end{cases}$$



T

term of a sequence

A term of a sequence is an individual number, figure, or letter in the sequence.

Example

In the sequence 2, 4, 6, 8, 10, the first term is 2, the second term is 4, and the third term is 6.

trinomial

Polynomials with exactly three terms are trinomials.

Example

The polynomial $5x^2 - 6x + 9$ is a trinomial.

V

vertex form

A quadratic function written in vertex form is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$.

Example

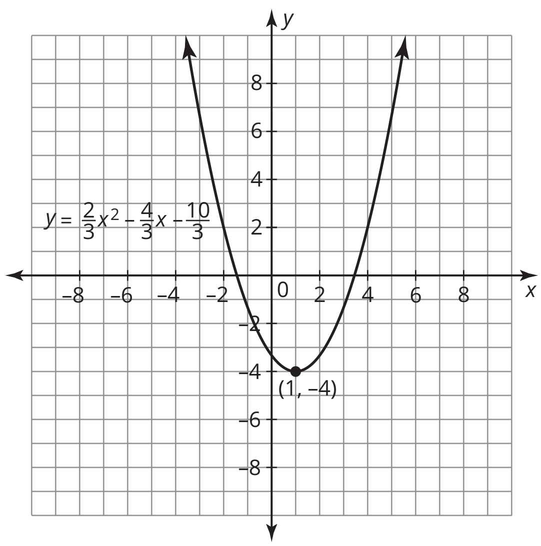
The quadratic equation $y = 2(x - 5)^2 + 10$ is written in vertex form. The vertex of the graph is the point (5, 10).

vertex of a parabola

The vertex of a parabola is the lowest or highest point on the graph of the quadratic function.

Example

The vertex of the graph of $y = \frac{2}{3}x^2 - \frac{4}{5}x - \frac{10}{3}$ is the point (1, −4), the absolute minimum of the parabola.

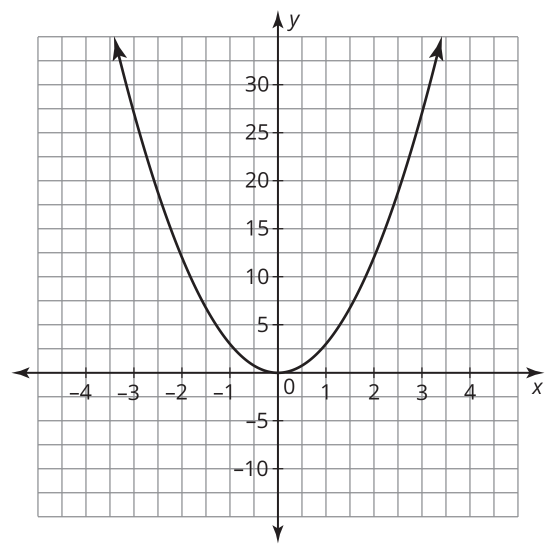


Vertical Line Test

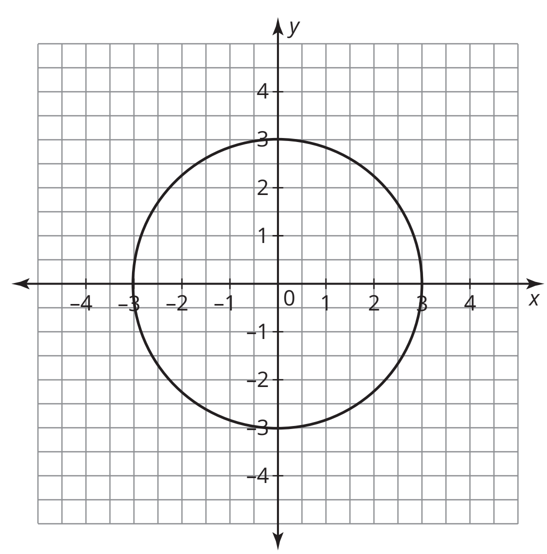
The Vertical Line Test is a visual method used to determine whether a relation represented as a graph is a function.

Example

The equation $y = 3x^2$ is a function. The graph passes the Vertical Line Test because there are no vertical lines that can be drawn that would intersect the graph at more than one point.



The equation $x^2 + y^2 = 9$ is not a function. The graph fails the Vertical Line Test because a vertical line can be drawn that intersects the graph at more than one point.



vertical motion model

A vertical motion model is a quadratic equation that models the height of an object at a given time. The equation is of the form $g(t) = -16t^2 + v_0t + h_0$, where $g(t)$ represents the height of the object in feet, t represents the time in seconds that the object has been moving, v_0 represents the initial velocity (speed) of the object in feet per second, and h_0 represents the initial height of the object in feet.

Example

A rock is thrown in the air at a velocity of 10 feet per second from a cliff that is 100 feet high. The height of the rock is modeled by the equation $y = -16t^2 + 10t + 100$.

X

x-intercept

The point where a graph crosses the x-axis is the x-intercept.

Y

y-intercept

The point where a graph crosses the y-axis is the y-intercept.

Z

zero of a function

A zero of a function is a real number that makes the value of the function equal to zero, or $f(x) = 0$.

Example

The zero of the linear function $f(x) = 2(x - 4)$ is (4, 0).

The zeros of the quadratic function $f(x) = -2x^2 + 4x$ are (0, 0) and (2, 0).

zero product property

The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

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